ARITHMETIC PARTIAL DIFFERENTIAL EQUATIONS

ALEXANDRU BUIUM AND SANTIAGO R. SIMANCA

Abstract. We develop an arithmetic analogue of linear partial differential equations in two independent "space-time" variables. The spatial derivative is a Fermat quotient operator, while the time derivative is the usual derivation. This allows us to "flow" integers or, more generally, points on algebraic groups with coordinates in rings with arithmetic flavor. In particular, we show that elliptic curves have certain canonical "flows" on them that are the arithmetic analogues of the heat and wave equations. The same is true for the additive and the multiplicative group.

1. Introduction

In this paper, we consider arithmetic partial differential equations in two "space-time" variables, a higher dimensional analogue of the theory of arithmetic ordinary differential equations developed in [5, 7, 9]. In the ordinary case, the rôle of functions of one variable is played by integers, and that of the derivative operator is played by a "Fermat quotient operator" with respect to a fixed prime p. Instead, we now take power series in a variable q with integer coefficients as the analogues of functions of two variables, and while maintaining the idea that a Fermat quotient type operator with respect to p is the analogue of the derivative in the "arithmetic direction," we now add the usual derivative with respect to q to play the rôle of a derivative in the "geometric direction." This leads to the study of some "arithmetic flows" of remarkable interest.

In the ordinary case [5, 7, 9] the "arithmetic direction" was viewed as a "temporal direction." In the present paper, the "arithmetic direction" is viewed as a "spatial direction," and the "geometric direction" is the "time." Under this interpretation, we will be able to think of points on algebraic varieties with coordinates in number theoretic rings as "functions of space," and we will be able to flow these points using the geometric parameter q, parameter that morally speaking plays the rôle of (the exponential of) time.

We proceed to explain our idea in some detail, and begin by discussing some of the basic aspects of evolution partial differential equations as they appear in classical analysis, discussion that out of necessity will be carried out in a non-rigorous fashion. In particular, the word "function" will be used to refer to functions (or even distributions) belonging to unspecified classes, and we will ignore all questions on convergence, as well as those concerning the proper definition of certain products or convolutions. Instead, we will concentrate exclusively on the formal aspects of the story, and examine only those concepts whose arithmetic analogue will later play a rôle in our study. We will then describe qualitatively what these arithmetic analogues are, and will end the introduction by a presentation of the basic problems and results of our theory.

1.1. **Evolution equations in analysis.** We denote by $\mathbb{R}_x$ the real line with "space" coordinate x and $\mathbb{R}_t$ the real line with "time" coordinate t. We let $\mathcal{F}(\mathbb{R}_x)$ be the ring of complex valued functions $f(x)$ on $\mathbb{R}_x$, and we let $\mathcal{F}(\mathbb{R}_x \times \mathbb{R}_t)$ be the ring of complex valued functions $v(x,t)$ on $\mathbb{R}_x \times \mathbb{R}_t$. Both of these rings are equipped with pointwise addition and multiplication. We will sometimes evaluate functions at complex values of t by "analytic continuation."

1.1.1. *Linear partial differential operators.* We consider first a general r-th order partial differential operator "in $1+1$ variables." This is just an operator of the form

$$\begin{array}{rcl} \mathcal{F}(\mathbb{R}_x \times \mathbb{R}_t) & \stackrel{P}{\to} & \mathcal{F}(\mathbb{R}_x \times \mathbb{R}_t) \\ Pu & := & P\left(x,t,u,Du,D^2u,\ldots,D^r u\right) , \end{array} \tag{1}$$

acting on functions $u=u(x,t)$. In this expression, $D^n u$ stands for the $n+1$ functions $\partial_x^{n-k}\partial_t^k u$, $0 \leq k \leq n$, where ∂_x and ∂_t are the corresponding partial derivative in the x and t directions, and $P(x,t,z)$ is a complex valued function of $\frac{(r+1)(r+2)}{2}+2$ complex variables. The operator is said to be *linear* if $P(x,t,z)$ is a linear function in the vector variable z. In that case, we define the *full symbol* $\sigma(P)(x,t,\xi,\tau)$ of P, a polynomial in (ξ,τ) with coefficients that are functions of (x,t), by replacing $\partial_x^{n-k}\partial_t^k u$ in P with the monomials $i^n\xi^{n-k}\tau^k$. Modulo terms of degree $r-1$ or less, $\sigma(P)$ is an invariantly defined function on the cotangent bundle $T^*(\mathbb{R}_x \times \mathbb{R}_t)$. If $P(x,t,z)$ is independent of (x,t), we say that the operator has constant coefficients. Standard examples of partial differential operators that are linear and have constant coefficients are

$$\begin{array}{rcll} Pu & = & \partial_t u - c\partial_x u\,, & \textit{the convection operator.} \\ Pu & = & \partial_t u - c\partial_x^2 u\,, & \textit{the heat operator,} \\ Pu & = & \partial_t^2 u - c\partial_x^2 u\,, & \textit{the wave operator,} \end{array} \tag{2}$$

respectively. Here, c is a constant that in the last two cases is assumed to be positive. These operators are of particular importance to us as their arithmetic analogues play a significant rôle in this article.

In regard to these examples, some remarks are in order:

(1) If in the heat operator, we replace the real parameter c by a purely imaginary constant, then we obtain the Schrödinger operator. As the results of this paper will suggest, our arithmetic analogue of the heat operator may also deserve attention as an analogue of the Schrödinger operator.
(2) If in the heat operator we interchange t and x, then we obtain the *sideways heat operator*. This operator will also have an analogue in our arithmetic theory.
(3) If in the wave operator we replace the positive constant c by a negative one, then we obtain the Laplace operator. Similarly, if in the convection operator c is replaced by a purely imaginary constant, we obtain the Cauchy-Riemann operator. These are the typical examples of elliptic operators, and they do not have analogues among the arithmetic partial differential operators discussed here.

Given a linear partial differential operator P, we may consider the *linear partial differential equation* $Pu=0$, and its *space of solutions*,

$$\mathcal{U} = \mathcal{U}_P := \{u \in \mathcal{F}(\mathbb{R}_x \times \mathbb{R}_t) : \; Pu = 0\}\,,$$

that is to say, the kernel of P. One of the main problems of the theory is to describe the structure of this space, and to describe the image of the linear mapping on functions defined by P. Typically, a function u in $\mathcal{U}$ will depend on ρ arbitrary functions $C_1(\xi), \dots, C_\rho(\xi)$, ρ some integer not exceeding the order r of P, while the description of the image of P deals with the *inhomogeneous equation* $Pu = \varphi$ for a given φ, and the set of conditions on it under which the said equation has a solution. By duality, the latter problem is usually reduced to the consideration of the space of solutions for the dual linear operator P^*.

1.1.2. *Exponential solutions and characteristic roots.* Assume P has constant coefficients and that u is in its space of solutions. The dependence of u on the functions $C_j(\xi)$ above is easily obtained by using Fourier transform, as follows. Let $\hat{u}(\xi, t)$ denote the Fourier transform of u in the x variable. Then the equation $Pu = 0$ yields

$$\sigma(P)(-\xi, -i\partial_t)\hat{u}(\xi, t) = 0 \,, \tag{3}$$

an ordinary differential equation in the variable t with parameter ξ. By solving this equation and applying the inverse Fourier transform in the parameter ξ, we obtain that

$$u(x,t) = \sum_{j=1}^{\rho} \int C_j(\xi) e^{-i\xi x - i\tau_j(\xi)t} d\xi \,, \tag{4}$$

for some functions $C_j(\xi)$, where, for each ξ, the numbers $\tau_1(\xi), \dots, \tau_\rho(\xi)$ are the *characteristic roots* of P, that is to say, the (complex) roots of the *characteristic polynomial* $\sigma(P)(-\xi, -\tau) \in \mathbb{R}[\tau]$, chosen to depend continuously on ξ. (We ignore here the problems arising from the possible presence of multiple roots.)

The "kernels" $e^{-i\xi x - i\tau_j(\xi)t}$ in (4) are the *exponential solutions* in the space of solutions for P, and the formula exhibits the general element of this space as a sum of ρ functions. There is one exponential solution per characteristic root, and the general solution u depends $\mathbb{C}$-linearly on one arbitrary function per root. (An important analytic aspect ignored here is that, in order to produce suitable distributional solutions through the formal manipulations above, we may need to choose some of the functions $C_j(\xi)$ to vanish identically. This is dictated by the behavior of the characteristic roots, and could make u depend on fewer than ρ arbitrary functions. For instance, think of the case on an elliptic operator, where some of the exponential solutions grow exponentially fast.)

We observe that for any ξ and τ, the exponentials $u_{\xi,\tau}(x,t) := e^{-i\xi x - i\tau t}$ "diagonalize" P. Indeed, we have

$$Pu_{\xi,\tau} = \sigma(P)(-\xi, -\tau) \cdot u_{\xi,\tau} \,. \tag{5}$$

This fact leads naturally to the study of the inhomogeneous equation $Pu = \varphi$ by way of Fourier inversion.

1.1.3. *Boundary value problem.* The classical approach to pinning down the functions $C_j(\xi)$ in (4) is by imposing "boundary conditions" on the solution of u. For suppose we have a ρ-tuple $B = (B_1, \dots, B_\rho)$ of linear partial differential operators $\mathcal{F}(\mathbb{R}_x \times \mathbb{R}_t) \overset{B_j}{\to} \mathcal{F}(\mathbb{R}_x \times \mathbb{R}_t)$. We consider the *restriction operator*

$$\begin{array}{ccc} \mathcal{F}(\mathbb{R}_x \times \mathbb{R}_t) & \overset{\gamma}{\to} & \mathcal{F}(\mathbb{R}_x) \\ \gamma v & := & v_{|t=0} \,, \end{array}$$

and if we let B^0_j stand for the composition $\gamma \circ B_j$, we obtain the *boundary value operator*

$$\begin{array}{rcl} \mathcal{F}(\mathbb{R}_x \times \mathbb{R}_t) & \stackrel{B^0}{\to} & \mathcal{F}(\mathbb{R}_x)^\rho \\ B^0 u & := & (B^0_1 u, \dots, B^0_\rho u)\,. \end{array}$$

We then say that the *boundary value problem for* (P, B) *is well posed* if for any $g \in \mathcal{F}(\mathbb{R}_x)^\rho$ there exists a unique element u in the space of solutions $\mathcal{U}_P$ whose boundary value $B^0 u$ is equal to g. In other words, the mapping

$$B^0_P : \mathcal{U}_P \to \mathcal{F}(\mathbb{R}_x)^\rho$$

given by the restriction of B^0 to $\mathcal{U}_P$ is a $\mathbb{C}$-linear isomorphism. (Classically, the domain and range are endowed with some topology, and the continuity of both, the mapping and its inverse, are also required; we ignore that consideration here.) In the case where P and B_j have constant coefficients, a formal computation shows that the functions $C_j(\xi)$ and $g_j(x) := B^0 u$ are related by the equalities

$$\sum_{k=1}^{\rho} \sigma(B_j)(-\xi, -\tau_k(\xi)) C_k(\xi) = \hat{g}_j(\xi), \quad 1 \le j \le \rho\,. \tag{6}$$

The determinant of the matrix of this system is the *Lopatinski determinant.* Its non-vanishing is "morally" equivalent to the well posedness condition; cf. [12], pp 321-322, or [17].

A classical choice for the operators B_j (corresponding to the *Cauchy problem*) is

$$B_j u = \partial_t^{j-1} u, \quad 1 \le j \le \rho\,. \tag{7}$$

For the classical operators P listed in (2) and the operators B_j in (7), the corresponding boundary value problems are all well posed.

1.1.4. *Propagator and Huygens principle.* Let us suppose that we have given operators $P, B_1, \dots, B_\rho$ such that the boundary value problem for (P, B) is well posed. Assume further that $P, B_1, \dots, B_\rho$ commute with the time translation operators

$$\begin{array}{c} \mathcal{F}(\mathbb{R}_x \times \mathbb{R}_t) \stackrel{L_{t_0}}{\to} \mathcal{F}(\mathbb{R}_x \times \mathbb{R}_t) \\ L_{t_0}(g(x,t)) = g(x, t + t_0) \end{array}$$

for all t_0. (This is the case, for instance, if the operators $P, B_1, \dots, B_\rho$ have constant coefficients.) Then $\mathcal{U}_P$ is stable under all L_t, and we have the $\mathbb{C}$-linear isomorphisms

$$B^t_P := B^0_P \circ L_t \, : \mathcal{U}_P \to \mathcal{F}(\mathbb{R}_x)^\rho\,,$$

explicitly given by

$$B^{t_0}_P u = (B_1 u, \dots, B_\rho u)_{|t=t_0}\,.$$

For any pair t_1 and t_2, we obtain the *evolution* or *propagator* operator, defined as the $\mathbb{C}$-linear isomorphism

$$S_{t_1,t_2} := B^{t_2}_P \circ (B^{t_1}_P)^{-1} = B^0_P \circ L_{t_2 - t_1} \circ (B^0_P)^{-1} : \mathcal{F}(\mathbb{R}_x)^\rho \to \mathcal{F}(\mathbb{R}_x)^\rho\,.$$

This family of operators satisfies the 1-parameter group property

$$S_{0, t_1 + t_2} = S_{0,t_1} \circ S_{0,t_2}\,,$$

a weak form of the "Huygens principle."

1.1.5. *Fundamental solutions.* The idea of evolution operator above is closely related to the concept of *fundamental solution.* In order to review this concept, let $\mathcal{F}_\star(\mathbb{R}_x)$ be the Abelian group $\mathcal{F}(\mathbb{R}_x)$, viewed as a ring with respect to convolution

$$(f \star g)(x) := \int f(y)g(x-y)dy\,.$$

We also let $\mathcal{F}_\star(\mathbb{R}_x \times \mathbb{R}_t)$ be the Abelian group $\mathcal{F}(\mathbb{R}_x \times \mathbb{R}_t)$, viewed as a module over $\mathcal{F}_\star(\mathbb{R}_x)$ with respect to convolution in the variable x. Then, if P commutes with the translation operators in the variable x (for instance, if P has constant coefficients), then the space of solutions $\mathcal{U}_P$ is a $\mathcal{F}_\star(\mathbb{R}_x)$-submodule of $\mathcal{F}_\star(\mathbb{R}_x \times \mathbb{R}_t)$. In this situation, we will assume further that B^0 is a $\mathcal{F}_\star(\mathbb{R}_x)$-module homomorphism. This is the case for the operators B_j in (7).

Under these circumstances, if the boundary value problem for (P, B) is well posed, B^0_P is an $\mathcal{F}_*(\mathbb{R}_x)$-module isomorphism so the space of solutions $\mathcal{U}_P$ is a free $\mathcal{F}_\star(\mathbb{R}_x)$-module of rank ρ with a unique basis

$$u_1(x,t), \dots, u_\rho(x,t) \tag{8}$$

(that generally speaking, will consist of distributions) such that the $\rho \times \rho$ matrix $(B^0_j u_i)$ is diagonal, with diagonal entries the Dirac delta function $\delta_0 = \delta_0(x)$ centered at 0. This basis is the *system of fundamental solutions* of (P, B). Clearly, for any $u \in \mathcal{U}_P$, we have that

$$u(x,t) = \sum_{i=1}^{\rho} (B^0_i u)(x) \star u_i(x,t)\,, \tag{9}$$

and so the fundamental solutions u_i appear as kernels in this integral representation for u. Conversely, let us assume that the operators B_j are as in (7), and that we can find $K_1, \dots, K_\rho \in \mathcal{U}_P$ such that any $u \in \mathcal{U}_P$ can be written uniquely as

$$u(x,t) = \sum_{i=1}^{\rho} (\partial_t^{i-1} u)(x,0) \star K_i(x,t)\,.$$

(Cf. [25], p. 138, for the case of the operators listed in (2).) Applying ∂_t^{j-1} to this identity, $1 \leq j \leq \rho$, and letting $t \to 0$, we get that the matrix $K = K(x,t)$, whose entries are given by $K_{ij} = \partial_t^{j-1} K_i$, $1 \leq i, j \leq \rho$, has the property that $K(x, 0)$ is equal to $\mathrm{diag}(\delta_0, \dots, \delta_0)$. Thus, we conclude that $K_1, \dots, K_\rho$ is the system of fundamental solutions of (P, B). The matrix K is the *fundamental solution matrix.*

For simplicity, let us assume further that $Pu = \partial_t^\rho u - c\partial_x^s u$. As K_ρ is in the space of solutions of P, we see that $\partial_t^{\rho-1} K_\rho, \partial_t^{\rho-2} K_\rho, \dots, \partial_t^0 K_\rho$ is also a system of fundamental solutions for (P, B). The uniqueness implies that

$$K_1 = \partial_t^{\rho-1} K_\rho,\ K_2 = \partial_t^{\rho-2} K_\rho,\ \dots,\ K_{\rho-1} = \partial_t K_\rho\,.$$

Notice that

$$S_{0,t}(g(x)) = g(x) \star K(x,t)$$

for any $g(x) \in \mathcal{F}(\mathbb{R}_x)^\rho$, that is, the propagator operator is given by convolution with the fundamental solution matrix.

It is worth recalling that K_ρ above may be used to solve the *inhomogeneous equation* $Pu = \varphi$, as in the following discussion where for simplicity we take once again $Pu = \partial_t^\rho u - c\partial_x^s u$. Cf. [17], pp. 80, 109, or [12], pp. 142, 235. Let $H \in \mathcal{F}(\mathbb{R}_t)$ be the characteristic function of the interval $[0, \infty)$ (the *Heaviside function*). If we

set $K_+(x,t) := K_\rho(x,t)\cdot H(t)$ where, of course, we are implicitly assuming that the product of the distributions in the right hand side is well defined, then we see that

$$PK_+(x,t) = \delta_0(x)\delta_0(t)\,.$$

A function of (x,t) satisfying this equation is said to be a *fundamental solution of the inhomogeneous equation.* A formal computation shows that for any $\varphi(x,t) \in \mathcal{F}(\mathbb{R}_x \times \mathbb{R}_t)$, the function $K_+ \star \varphi$ (where $\star$ denotes now the convolution with respect to both variables x and t,) is a solution to the equation $Pu = \varphi$, that is to say,

$$P(K_+ \star \varphi) = \varphi\,.$$

1.2. **Evolution equations in arithmetic.** The main purpose of this paper is to propose an arithmetic analogue of the "1 + 1 evolution picture" above. In the remaining portion of this introduction, we informally present our main concepts, problems, and results.

1.2.1. *Main concepts.* In our arithmetic theory, the ring $\mathcal{F}(\mathbb{R}_x)$ of functions in x is replaced by a "ring of numbers" R. A natural choice [5, 7, 9] for this R is given by the completion of the maximum unramified extension of the ring $\mathbb{Z}_p$ of p-adic integers. We will think of p as a "space variable," the analogue of x. And the analogue of the ring $\mathcal{F}(\mathbb{R}_t \times \mathbb{R}_x)$ of functions of space-time is the ring of formal power series $A = R[[q]]$, whose elements are viewed as "superpositions" of the "plane waves" aq^n, $a \in R$, analogues of the plane waves $a(x)e^{-2\pi int}$ of frequency n. (We will use other rings also, for instance, $R[[q^{-1}]]$. Series in $R[[q]]$ will be viewed as superpositions of plane waves "involving non-negative frequencies only," whereas series in $R[[q^{-1}]]$ will be superpositions of plane waves "involving non-positive frequencies only." It will be interesting to further enlarge these by considering the rings of Laurent power series and their p-adic completions, $R((q))\widehat{\ }$ and $R((q^{-1}))\widehat{\ }$, respectively.)

The rôle of the partial derivative $-(2\pi i)^{-1}\partial_t$ is to be played by the derivation

$$\begin{array}{ccc} A & \stackrel{\delta_q}{\to} & A \\ \delta_q u & := & q\partial_q u \end{array}\,, \tag{10}$$

where ∂_q is the usual derivative with respect to q. On the other hand, the analogue of the partial derivative $(2\pi i)^{-1}\partial_x$, which should be interpreted as a derivative with respect to the prime p, is obtained by following the idea in [5, 7, 9]. Indeed, we propose to define this derivative with respect to p as the "Fermat quotient operator" given by

$$\begin{array}{ccc} A & \stackrel{\delta_p}{\to} & A \\ \delta_p u & := & \dfrac{u^{(\phi)}(q^p) - u(q)^p}{p} \end{array}\,, \tag{11}$$

where the upper index (ϕ) stands for the operation of twisting the coefficients of a series by the unique automorphism $\phi : R \to R$ that lifts the p-th power Frobenius automorphism of R/pR. Note that the restriction of δ_p to R is the mapping

$$\begin{array}{ccc} R & \stackrel{\delta_p}{\to} & R \\ \delta_p a & = & \dfrac{\phi(a) - a^p}{p} \end{array}\,, \tag{12}$$

which is the arithmetic analogue of a derivation, as discussed in [5, 7, 9]. Note that the set of its *constants*, $R^{\delta_p} := \{a \in R; \delta_p a = 0\}$ consists of 0 and the roots of unity in R. So R^{δ_p} is a multiplicative monoid but not a subring of R.

In order to proceed, we need to describe the analogues of linear partial differential equations. We start, more generally, with maps of the form

$$\begin{array}{rcl} A & \stackrel{P}{\rightarrow} & A \\ Pu & := & P(u, Du, \ldots, D^r u), \end{array} \tag{13}$$

where $P = P(z)$ is a p-adic limit of polynomials with coefficients in A in $\frac{(r+1)(r+2)}{2}$ variables, and $D^n u$ stands for the $n+1$ series $\delta_p^i \delta_q^{n-i} u$, $0 \le i \le n$. These maps are the "partial differential operators" of order r in this article. What remains to be done is to define the notion of linearity for them.

The naive requirement that $P(z)$ be a linear form in z is not appropriate. Indeed, the property that linearity should really capture is that differences of solutions be again solutions, and this is not going to happen since δ_p itself is not additive. We could insist upon asking that the map $u \mapsto Pu$ be additive, but as such, this condition would lead to a rather restricted class of examples. In order to find what we suggest is the right concept (which, in particular, will provide sufficiently many interesting examples), we proceed by generalizing our setting as follows (cf. [5, 7, 9] for the ordinary differential case).

Firstly, we consider mappings $A^N \to A$ as in (13), where now u is an N-tuple. Secondly, we restrict such maps to subsets $X(A) \subset \mathbb{A}^N(A) = A^N$, where $X \subset \mathbb{A}^N$ is a closed subscheme of the affine N-space $\mathbb{A}^N$ over A, and where $X(A)$ denotes the set of points of X with coordinates in A. If X has relative dimension n over A, the induced maps $X(A) \to A$ will be viewed as "partial differential operators" on X in $1+1$ "independent variables" and n "dependent variables." Using a gluing procedure, we then derive the concept of a "partial differential operator," $X(A) \to A$, on (the set of A-points of) an arbitrary scheme X of finite type over A (which need not be affine). Finally, when we take X in this general set-up to be a commutative group scheme G over A, we define a *linear partial differential operator* on G to be a "partial differential operator" $G(A) \to A$ on G that is also a group homomorphism, where A is viewed with its additive group structure. By making this "set-theoretical" definition one that is "scheme-theoretical" (varying A), we arrive at the notion of *linear partial differential operator* that we propose in here. And once again, we are able to associate to any linear partial differential operator so defined a polynomial $\sigma(\xi_p, \xi_q)$ in two variables with A-coefficients, which we refer to as the (*Picard-Fuchs*) *symbol* of the operator.

The construction above is motivated by points of view adopted in analysis and mathematical physics. Indeed, we view $\operatorname{Spec} A$ as the analogue of $\mathbb{R}_x \times \mathbb{R}_t$, and we view schemes X as the analogues of manifolds M, so the set $X(A) = \operatorname{Hom}(\operatorname{Spec} A, X)$ is the analogue of the set $\mathcal{F}(\mathbb{R}_x \times \mathbb{R}_t, M)$ of maps $\mathbb{R}_x \times \mathbb{R}_t \to M$ (which we require here to be at least continuous). The commutative group schemes G of relative dimension 1 (which will be the main concern of this paper) are the analogues of Lie groups of the form $\mathbb{C}/\Gamma$ where Γ is a discrete subgroup of $\mathbb{C}$. (There are 3 cases, those corresponding to a subgroup Γ of rank 0, 1, or 2 respectively. The corresponding groups $\mathbb{C}/\Gamma$ are the additive group $\mathbb{C}$, the multiplicative group $\mathbb{C}^\times$, and the elliptic curves E over $\mathbb{C}$) On the other hand, any linear partial differential operator $\mathcal{F}(\mathbb{R}_x \times \mathbb{R}_t) \to \mathcal{F}(\mathbb{R}_x \times \mathbb{R}_t)$ with symbol that vanishes at $(0,0)$ induces a

homomorphism

$$\mathcal{F}(\mathbb{R}_x \times \mathbb{R}_t)/\Gamma \to \mathcal{F}(\mathbb{R}_x \times \mathbb{R}_t)\,.$$

We can consider then the composition

$$\mathcal{F}(\mathbb{R}_x \times \mathbb{R}_t, \mathbb{C}/\Gamma) \simeq \mathcal{F}(\mathbb{R}_x \times \mathbb{R}_t)/\Gamma \to \mathcal{F}(\mathbb{R}_x \times \mathbb{R}_t). \tag{14}$$

Our groups $G(A)$ are the analogues of the groups $\mathcal{F}(\mathbb{R}_x \times \mathbb{R}_t, \mathbb{C}/\Gamma)$, and our linear partial differential operators $G(A) \to A$ are the analogues of the operators in (14).

Given a linear partial differential operator $G(A) \stackrel{P}{\to} A$ in the arithmetic setting, we may consider the *group of solutions* consisting of those $u \in G(A)$ such that $Pu = 0$. This is a subgroup of $G(A)$. Those elements in this space that "do not vary with time" will define the concept of *stationary solutions*. Notice that if G descends to R, *stationary* will simply mean "belonging to $G(R)$."

The linear partial differential operators in the sense above will be called $\{\delta_p, \delta_q\}$-*characters*, and we will denote them by ψ rather than P. This will put our terminology and notation in line with that used in [5, 7, 9], where the case of arithmetic ordinary differential equations was treated.

As a matter of fact, some comments on the ordinary case are in order here. If all throughout the theory sketched above we were to insist that the operators $u \mapsto Pu$ in (13) be given by polynomials in $u, \delta_q u, \ldots, \delta_q^r u$ only, we would then be led to the Ritt-Kolchin theory of "ordinary differential equations" with respect to δ_q, cf. [26, 20, 11]. In particular, this would lead to the notion of δ_q-*character* of an algebraic group, which in turn should be viewed as the analogue of a linear ordinary differential operator on an algebraic group (cf. to the Kolchin logarithmic derivative of algebraic groups defined over R, [20, 11], and the Manin maps of Abelian varieties defined over $R[[q]]$, [22, 2]).

If on the other hand we were to insist throughout the theory that the operators $u \mapsto Pu$ in (13) be given by p-adic limits of polynomials in $u, \delta_p u, \ldots, \delta_p^r u$ only, we would then be led to the arithmetic analogue of the ordinary differential equations in [5, 7, 9]. In particular, we would then arrive at the notion of a δ_p-character of a group scheme, which is the arithmetic analogue of a linear ordinary differential equation on a group scheme.

1.2.2. *Main problems.* We present, in what follows, a sample of the main problems to be treated in this paper:

1. Find all $\{\delta_p, \delta_q\}$-characters on a given group scheme G.
2. For any $\{\delta_p, \delta_q\}$-character ψ, describe the kernel of ψ, that is to say, the group of solutions of $\psi u = 0$, and study the behavior of the solutions in terms of "convolution," "boundary value problems," "characteristic polynomial," "propagators," "Huygens' principle," etc.
3. For any $\{\delta_p, \delta_q\}$-character ψ, describe the image of ψ, that is to say, the group of all the φ such that the inhomogeneous equation $\psi u = \varphi$ has a solution.

These will be discussed in detail for the case of one dependent variable, that is to say, for groups of dimension one. Indeed, we shall thoroughly analyze the additive group $G = \mathbb{G}_a$, the multiplicative group $G = \mathbb{G}_m$, and elliptic curves $G = E$ over A, cases where $G(A)$ corresponds to the additive group $(A, +)$, the multiplicative group $(A^\times, \cdot)$ of invertible elements of A, and the group $(E(A), +)$ of points with

coordinates in A of a projective non-singular cubic, with addition operation given by the chord-tangent construction, respectively.

1.2.3. *Main results.* In regard to Problem 1, we will start by proving that the $\{\delta_p, \delta_q\}$-characters of fixed order on a fixed group scheme form a finitely generated A-module. We will then provide a rather complete picture of the space of $\{\delta_p, \delta_q\}$-characters in the cases where G is either $\mathbb{G}_a$, or $\mathbb{G}_m$, or an elliptic curve E over A. In particular, it will turn out that these groups possess certain remarkable $\{\delta_p, \delta_q\}$-characters that are, roughly speaking, the analogues of the classical operators listed in (2) above.

In the cases where G is either $\mathbb{G}_a$, or $\mathbb{G}_m$, or an elliptic curve E defined over R, the $\{\delta_p, \delta_q\}$-characters ψ of G are "essentially" built from δ_q-characters ψ_q, and δ_p-characters ψ_p of G. This situation is analogous to the one in classical analysis in $\mathbb{R}_x \times \mathbb{R}_t$, where linear differential operators are "built" from ∂_x and ∂_t, respectively. Remarkably, however, if G is an elliptic curve E over A that is sufficiently "general," then E possesses a $\{\delta_p, \delta_q\}$-character ψ^1_{pq} that cannot be built from δ_q- and δ_p-characters alone. Thus, from a global point of view, this $\{\delta_p, \delta_q\}$-character ψ^1_{pq} is a "pure partial differential object," in the sense that it cannot be built from "global ordinary linear differential objects."

This will all unravel in the following manner. We will first prove that for any elliptic curve E over A, there always exists a non-zero $\{\delta_p, \delta_q\}$-character ψ^1_{pq} of order one[1]. We will then show that for a general elliptic curve E over A, the A-module of $\{\delta_p, \delta_q\}$-characters of order one has rank one, and so ψ^1_{pq} is essentially unique. We view it as a *canonical convection equation* on the elliptic curve. Let us note that ψ^1_{pq} cannot be decomposed as a linear combination of δ_q- and δ_p-characters alone because, on these elliptic curves, all such characters of order one are trivial.

The $\{\delta_p, \delta_q\}$-character ψ^1_{pq} will turn out to be a factor of a canonical order two $\{\delta_p, \delta_q\}$-character that can be expressed as $\psi^2_q + \lambda\psi^2_p$, the sum of a δ_q-character ψ^2_q of order two, and of λ times a δ_p-character ψ^2_p of order two also, $\lambda \in A$. We will view this $\{\delta_p, \delta_q\}$-character of order two as a *canonical wave equation* on the elliptic curve.

Once again, for a general elliptic curve E over R, we will encounter *heat equations* on E also. These will be sums of the form $\psi^1_q + \lambda\psi^2_p$, where ψ^1_q is a δ_q-character of order one, ψ^2_p is a δ_p-character of order two, and $\lambda \in R$.

In regard to Problem 2, a first remark is that the "generic" $\{\delta_p, \delta_q\}$-characters do not admit non-stationary solutions hence a natural question is to characterize those that admit such solutions. We succeed in giving such a characterization under a mild non-degeneracy condition on the symbol $\sigma(\xi_p, \xi_q)$, a condition that is satisfied "generically." Roughly speaking, we will show that a non-degenerate $\{\delta_p, \delta_q\}$-character ψ of a group G over R has non-stationary solutions if, and only if, the polynomial $\sigma(0, \xi_q)$ has an integer root. The effect of this criterion is best explained if we consider families ψ_λ of $\{\delta_p, \delta_q\}$-characters of low order (usually 1 or 2) that depend linearly on a parameter $\lambda \in R$, and ask for the values of this parameter for which ψ_λ possesses non-stationary solutions. We then discover a

[1]This is in deep contrast with the "ordinary case," where for a general E over A, there are no non-zero δ_q-characters of order one [22, 4], or where for a general E over A, or a general E over R, there are no non-zero δ_p-characters of order one [5, 9]. Here, an elliptic curve E over a ring is said to be *general* if the coefficients of the defining cubic belong to the ring and do not satisfy a certain "(arithmetic) differential equation."

"quantization" phenomenon according to which, the only values of λ for which this is so form a "discrete" set parameterized by integers $\kappa \in \mathbb{Z}$. This singles out certain $\{\delta_p, \delta_q\}$-characters as "canonical equations" on our groups, and produces, for instance, a *canonical heat equation* on a general elliptic curve over R.

A different kind of "quantization" will be encountered in our study of Tate curves with parameter βq, where $\beta \in R$. (These curves are defined over $R((q))$ but not over R.) In that case, we obtain that the canonical convection equation has non-stationary solutions if and only if the values of β are themselves "quantized," that is to say, parameterized by integers $\kappa \in \mathbb{Z}$.

The criterion above on the existence of non-stationary solutions will be a consequence of a more detailed analysis of spaces of solutions of $\psi u = 0$. In order to explain this, we take a non-degenerate $\{\delta_p, \delta_q\}$-character ψ of a group G over R, and consider first the groups $\mathcal{U}_{\pm 1}$ of solutions of $\psi u = 0$ in $G(A)$ ($A = R[[q^{\pm 1}]]$) vanishing at $q^{\pm 1} = 0$, respectively. Intuitively, these are the analogues of those solutions in analysis that "decay to zero" as $t \to \mp i\infty$, respectively. We will prove that $\mathcal{U}_{\pm 1}$ are finitely generated free R-modules under a natural convolution operation denoted by $\star$. The ranks of these R-modules are given by the cardinalities ρ_+ and ρ_- of the sets $\mathcal{K}_\pm$ of positive and negative integer roots of the polynomial $\sigma(0, \xi_q)$, where $\sigma(\xi_p, \xi_q)$ is the symbol of ψ. The integers in $\mathcal{K}_\pm$ are the *characteristic integers* of ψ.

For instance, if ψ is the arithmetic analogue of the convection or heat equation then one of the spaces $\mathcal{U}_{\pm 1}$ is zero and the other has rank one over R under convolution. If ψ is the arithmetic analogue of the wave equation then both spaces $\mathcal{U}_{\pm 1}$ have rank one over R, an analogue of the picture in d'Alembert's formula where one has 2 waves traveling in opposite directions.

Going back to the general situation of a non-degenerate $\{\delta_p, \delta_q\}$-character ψ, we will consider a *boundary value problem at* $q^{\pm 1} = 0$ as follows. Firstly, we will consider the operators

$$\begin{array}{rcl} R[[q^{\pm 1}]] & \stackrel{\Gamma_\kappa}{\to} & R \\ \Gamma_\kappa(\sum a_n q^n) & := & a_\kappa = \frac{1}{\kappa!}(\partial^\kappa_{q^{\pm 1}} u)_{|q^{\pm 1}=0} \end{array},$$

where, of course, $\partial_{q^{-1}} := -q^2 \partial_q$. Secondly, we note that, up to an invertible element in R, there is a unique non-zero δ_q-character ψ_q of G of minimal order. The $\{\delta_p, \delta_q\}$-character ψ_q has order 0 if $G = \mathbb{G}_a$, and order 1 if G is either $\mathbb{G}_m$ or an elliptic curve over R; in the latter case ψ_q is the Kolchin logarithmic derivative. For $\kappa \in \mathcal{K}_\pm$ we denote by B^0_κ the composition $\Gamma_\kappa \circ \psi_q$. We will then prove that the *boundary value operator at* $q^{\pm 1} = 0$,

$$\begin{array}{rcl} \mathcal{U}_{\pm 1} & \stackrel{B^0_\pm}{\to} & R^{\rho_\pm} \\ B^0_\pm u & = & (B^0_\kappa u)_{\kappa \in \mathcal{K}_\pm} \end{array},$$

is an R-module isomorphism, and, furthermore, that there exist solutions $u_\kappa \in \mathcal{U}_\pm$ such that for any $u \in \mathcal{U}_{\pm 1}$ we have the formula

$$u = \sum_{\kappa \in \mathcal{K}_\pm} (B^0_\kappa u) \star u_\kappa \,. \tag{15}$$

This can be viewed as analogue of the expression (4) because it exhibits u as a sum of $\rho_\pm$ terms indexed by the characteristic integers, with each term depending $\mathbb{Z}$-linearly on one arbitrary "function of space." It can also be viewed as an analogue

of (9) because of its formulation in terms of convolution. Then we interpret the bijectivity of $B^0_{\pm}$ by saying that the "boundary value problem at $q^{\pm 1}=0$" is *well posed*. (The language chosen here is a bit lax since no direct analogue of this boundary value problem at $q^{\pm 1}=0$ seems to be available in real analysis; indeed, such an analogue would prescribe boundary values at (complex) infinity, which does not appear as a natural condition to be imposed on solutions of linear partial differential equations in analysis.)

The solutions u_κ ($\kappa \in \mathcal{K}$) will be referred to as *basic solutions* of the $\{\delta_p, \delta_q\}$-character ψ. In some sense, these elements of the kernel of ψ are the analogues of both the exponential solutions and the fundamental solutions of the homogeneous equations in real analysis. Of course, these analogies have significant limitations.

An interesting feature of the solutions u in the kernel of ψ is the following "algebraicity modulo p" property. Let us denote by k the residue field of R, hence $k = R/pR$. Then, for any such u, the reduction modulo p, $\overline{\psi_q u} \in k[[q^{\pm 1}]]$, of the series $\psi_q u$ is integral over the polynomial ring $k[q^{\pm 1}]$, and the field extension

$$k(q) \subset k(q, \overline{\psi_q u})$$

is Abelian with Galois group killed by p. On the other hand, under certain general assumptions that are satisfied, in particular, by our "canonical" equations, we will prove that the solutions $u \neq 0$ of $\psi u = 0$ are transcendental over $R[q]$. This transcendence result can be viewed as a (weak) incarnation of Manin's Theorem of the Kernel [22].

Some of the results above about groups G over R have analogues for groups not defined over R, such as the Tate curves. For the latter the δ_q-character ψ_q will now be the Manin map, which has order 2.

If we instead consider solutions that do not necessarily decay to 0 as $q^{\pm 1} \to 0$, we are able to show that, in case $\rho_+ = 1$, the appropriate boundary value problem at $q \neq 0$ is well posed. (The condition $\rho_+ = 1$ is usually satisfied by our "canonical equations".) Roughly speaking, for groups G over R and $\{\delta_p, \delta_q\}$-characters ψ with $\rho_+ = 1$, this boundary value problem has the following meaning. We consider $q_0 \in pR^\times$ and $g \in G(R)$. If $A = R[[q]]$, we ask if there exists a (possibly unique) solution $u \in G(A)$ of the equation $\psi u = 0$ that satisfies the condition $B^{q_0} u = g$, where

$$\begin{array}{ccc} G(A) & \stackrel{B^{q_0}}{\to} & G(R) \\ B^{q_0} u & := & u(q_0) \end{array}$$

is the group homomorphism induced by the ring homomorphism $A = R[[q]] \to R$ that sends q into q_0. We view B^{q_0} as a *boundary value operator at* q_0, and we view the corresponding embedding $\operatorname{Spec} R \to \operatorname{Spec} A$ as the "curve" $q = q_0$ in the (p, q)-plane, along which we are given our boundary values.

Our point of view here is reminiscent of that in real analysis, where we replace real time by a complex number whose real part is small in order to avoid singularities on the real axis. Indeed the condition "q_0 invertible" (that is to say, $v_p(q_0) = 0$, where v_p is the p-adic valuation) is an analogue of the condition "real time;" the condition "q_0 non-invertible with small $v_p(q_0)$" is an analogue of "complex time close to the real axis;" "q_0 close to 0" (that is, $v_p(q_0)$ big) is an analogue of "time close to $-i\infty$."

For equations with $\rho_+ = 1$, we will investigate also the "limit of solutions as $q \to 0$" (or intuitively, as $t \to -i\infty$). This limit will sometimes exist, and if so, the

limit will usually be a torsion point of $G(R)$. In this sense, torsion points tend to play the role of "equilibrium states at (complex) infinity."

These principles do not hold uniformly in all examples. For instance, in the case of elliptic curves E over R, we will need to replace $E(R)$ by a suitable subgroup of it, $E'(R)$, in order to avoid points whose orders are powers of p. And for elliptic curves over A, rather than those over R (such as the Tate curves), the boundary value problem at $q \neq 0$ will take a slightly different form.

Once the boundary value problem at $q \neq 0$ has been solved, we can construct *propagators* as follows. Again, we fix ψ with $\rho_+ = 1$, choose "complex times" $q_1, q_2 \in pR^\times$, and consider the endomorphism S_{q_1,q_2} of the group $G(R)$ that sends any $g_1 \in G(R)$ into $g_2 := u(q_2)$, where $u \in G(A)$ is the unique solution to the boundary value problem

$$\begin{array}{rcl} \psi u & = & 0 \\ u(q_1) & = & g_1 \end{array} .$$

This construction does not work uniformly in all examples the way it is described here. In order to make it so, we need an appropriate modification of the given recipe. Nevertheless, in all situations, we find that the propagator S satisfies a 1-parameter group property. For it turns out that given "complex times" $q_i = \zeta_i q_0$ with $\zeta_i \in R$ a root of unity, $i = 1, 2$, we have that

$$S_{q_0,\zeta_1\zeta_2 q_0} = S_{q_0,\zeta_2 q_0} \circ S_{q_0,\zeta_1 q_0} ,$$

identity that can be viewed as a weak incarnation of Huygens' principle.

In regard to Problem 3, we will consider non-degenerate $\{\delta_p, \delta_q\}$-characters ψ of G, and we will find sufficient conditions on a given series $\varphi \in A = R[[q]]$ ensuring that the inhomogeneous equation $\psi u = \varphi$ have a solution $u \in G(A)$. Specifically, let us define the support of the series $\varphi = \sum c_n q^n$ as the set $\{n \, : \, c_n \neq 0\}$. We will then prove that if φ has support contained in the set $\mathcal{K}'$ of *totally non-characteristic* integers, then the inhomogeneous equation above, with φ as right hand side, has a unique solution $u \in G(A)$ for which the support of $\psi_q u$ is disjoint from the set $\mathcal{K}$ of characteristic integers. Here, $\mathcal{K}'$ is defined by an easy congruence involving the symbol, and, as suggested by the terminology, $\mathcal{K}'$ is disjoint from $\mathcal{K}$. Furthermore, if $\bar{\varphi}$, the reduction mod p of φ, is a polynomial, then we will prove a corresponding "algebraicity mod p" property for u stating that the series $\overline{\psi_q u}$ is integral over $k[q^{\pm 1}]$, and that the field extension $k(q) \subset k(q, \overline{\psi_q u})$ is Abelian with Galois group killed by p. On the other hand, for a "canonical" ψ and for a φ with *short* support, we will show that the solutions of $\psi u = \varphi$ are transcendental over $R[q]$.

The idea behind the results above is to construct, for all integers κ coprime to p, a solution u_κ of the equation

$$\psi u_\kappa = \frac{\sigma(0,\kappa)}{p} \cdot q^\kappa , \tag{16}$$

which we shall call a *basic* solution of the inhomogeneous equation. For κ a characteristic integer, the right hand side of (16) vanishes and our basic solutions are the previously mentioned basic solutions of the homogeneous equation. For κ a totally non-characteristic integer, the right hand side of (16) is a unit in R times q^κ, and that leads to the desired result about the inhomogeneous equation. Notice that (16) should be viewed as an analogue of (5), and the u_κ's (for $\kappa \in \mathbb{Z} \backslash p\mathbb{Z}$) should be viewed as (partially) diagonalizing ψ.

1.3. **Concluding remarks.** It is natural to ask for an extension of the theory in the present paper to the case of $d+e$ independent variables and n dependent variables, that is to say, to the case of d time variables $q_1, \dots, q_d$, e space variables $p_1, \dots, p_e$, where p_i are prime numbers, and groups G of (relative) dimension n. It is not hard to perform such an extension to the case $e = 1$, $d \geq 1$, $n \geq 1$ (that is to say, one prime p as space variable, $d \geq 1$ indeterminates q_i as time variables, and groups of dimension $n \geq 1$). In fact, all the difficulties of this more general case are already present in ours here, where $d = e = n = 1$. On the other hand, there seems to be no obvious way to extend the theory in a non-trivial way even to the case $d = 1$, $e > 1$, $n = 1$ (that is to say, two or more primes p_i as space variables, one time variable q, and groups of dimension 1). The main obstruction lies in the fact that when at least two primes are made to interact in the same equation, the solutions exhibit a rather fundamental divergent form.

We end our discussion here by summarizing (cf. the tables below) some of the similarities and differences between the set-ups of classical real analysis [17, 13, 25], classical p-adic analysis [10, 19], and arithmetic (in the spirit of [5, 7, 9] for the ordinary differential case, and the present paper for the partial differential case). For the ordinary differential case we have:

	real analysis	p-adic analysis	arithmetic
1-dimensional manifold	$\mathbb{R}_x$	R	R^{δ_p}
ring of functions	$\mathcal{F}(\mathbb{R}_x)$	$R[[x]]$	R
operator on functions	∂_x	∂_x	δ_p

For the partial differential case we have:

	real analysis	p-adic analysis	arithmetic
2-dimensional manifold	$\mathbb{R}_t \times \mathbb{R}_x$	$R \times R$	$R^{\delta_p} \times R$
ring of functions	$\mathcal{F}(\mathbb{R}_x \times \mathbb{R}_t)$	$R[[x,t]]$	$R[[q]]$
operators on functions	∂_x, ∂_t	∂_x, ∂_t	δ_p, δ_q

In the last column, the set R^{δ_p} plays a role similar to that of the set of "geometric points of the spectrum of the field with one element" in the sense of Deninger, Kurokawa, Manin, Soulé, and others [21, 23, 30]. (For comments on differences between our approach here and the ideology of the "field with one element," we refer to the Introduction of [9].) In particular R^{δ_p} can be viewed as an object of dimension zero. Notice that the third column appears to be obtained from the second one by "decreasing dimensions by one;" this reflects the fact that, in contrast to the second column, the third one treats numbers as functions. Note also that the interpretation of R^{δ_p} as a space of dimension zero is morally consistent with the (otherwise) curious fact that the groups of solutions $\mathcal{U}_1$ and $\mathcal{U}_{-1}$ are modules over R with respect to a "convolution" operation $\star$. Indeed, this suggests that "pointwise" multiplication and "convolution" of functions on R^{δ_p} coincide, which in turn, is compatible with viewing R^{δ_p} as having dimension 0.

1.4. **Plan of the paper.** We begin in §2 by introducing our main concepts, and where, in particular, we define $\{\delta_p, \delta_q\}$-characters, their various spaces of solutions, and the convolution module structure on these spaces. In §3 and §4, we construct and study partial differential jet spaces of schemes and formal groups, respectively. These are arithmetic-geometric analogues of the standard jet spaces of differential geometry, arithmetic in the space direction, and geometric in the time direction. The geometry of these jet spaces controls the structure of the spaces of $\{\delta_p, \delta_q\}$-characters. Among several other constructions, we define in §4 the Picard-Fuchs symbol of a $\{\delta_p, \delta_q\}$-character. In §5, we develop an analogue of the Fréchet derivative, and use it to define the Fréchet symbol of a $\{\delta_p, \delta_q\}$-character. We then establish a link between the Fréchet symbol and the Picard-Fuchs symbol that will be useful in applications. We also develop in this section a brief analogue of the Euler-Lagrange formalism. In §6, §7 and §8, we present our main results about $\{\delta_p, \delta_q\}$-characters and their solution spaces in the cases of $\mathbb{G}_a$, $\mathbb{G}_m$, and elliptic curves, respectively.

Acknowledgements. We would like to acknowledge useful conversations with F. J. Voloch and D. Thakur on the subject of transcendence.

2. Main concepts

In this section, we begin by introducing the main algebraic concepts in our study, especially the $\{\delta_p, \delta_q\}$-rings and $\{\delta_p, \delta_q\}$-prolongation sequences. These are then used to define $\{\delta_p, \delta_q\}$-morphisms of schemes, and, eventually, $\{\delta_p, \delta_q\}$-characters of group schemes. We introduce various solution spaces of $\{\delta_p, \delta_q\}$-characters, and discuss the convolution module structure on them.

Let p be a prime integer that we fix throughout the entire paper. For technical purposes related to the use of logarithms of formal groups, we need to assume that $p \neq 2$. Later on, in our applications to elliptic curves, we will need to assume that $p \neq 3$ also. All throughout, A shall be a ring, and B an algebra over A. If x is an element of A, we shall denote by x its image in B also.

We let $C_p(X,Y)$ stand for

$$C_p(X,Y) := \frac{X^p + Y^p - (X+Y)^p}{p}\,,$$

an element of the polynomial ring $\mathbb{Z}[X,Y]$.

After [5], we say that a map $\delta : A \to B$ is a *p-derivation* if $\delta_p(1) = 0$, and

$$\begin{array}{rcl} \delta_p(x+y) & = & \delta_p x + \delta_p y + C_p(x,y)\,, \\ \delta_p(xy) & = & x^p \delta_p y + y^p \delta_p x + p\delta_p x \delta_p y\,, \end{array}$$

for all $x, y \in A$, respectively. We will always write $\delta_p x$ instead of $\delta_p(x)$.

Given a δ_p-derivation, we define $\phi := \phi_p := \phi_{\delta_p} : A \to B$ by

$$\phi_p(x) = x^p + p\delta_p x\,, \tag{17}$$

a map that turns out to be a homomorphism of rings. Sometimes, we will write x^ϕ instead of $\phi_p(x)$, and when omitted from the notation, the context will indicate the p-derivation δ_p that is being used.

We recall that a map $\delta_q : A \to B$ is said to be a *derivation* if

$$\begin{array}{rcl} \delta_q(x+y) & = & \delta_q x + \delta_q y\,, \\ \delta_q(xy) & = & x\delta_q y + y\delta_q x\,, \end{array}$$

for all $x, y \in A$, respectively. For the time being, the index q will not be given any interpretation. Later on, we will encounter the situation where q is an element of A, and in that case, we will think of δ_q as a derivation in the "direction" q.

Definition 2.1. Let $\delta_p : A \to B$ be a p-derivation. A derivation $\delta_q : A \to B$ is said to be a δ_p-*derivation* if

$$\delta_q \delta_p x = p\delta_p\delta_q x + (\delta_q x)^p - x^{p-1}\delta_q x \tag{18}$$

for all $x \in A$.

In particular, for a δ_p-derivation $\delta_q : A \to B$, we have that

$$\delta_q \circ \phi_p = p \cdot \phi_p \circ \delta_q \, . \tag{19}$$

Conversely, if p is a non-zero divisor in A, then (19) implies (18).

Definition 2.2. A $\{\delta_p, \delta_q\}$-*ring* A is one equipped with a p-derivation $\delta_p : A \to A$ and a δ_p-derivation $\delta_q : A \to A$. A *morphisms of* $\{\delta_p, \delta_q\}$-*rings* is a ring homomorphism that commutes with δ_p and δ_q. A $\{\delta_p, \delta_q\}$-ring B is said to be a $\{\delta_p, \delta_q\}$-*ring over the* $\{\delta_p, \delta_q\}$-*ring* A if it comes equipped with a $\{\delta_p, \delta_q\}$-ring homomorphism $A \to B$. We say that a $\{\delta_p, \delta_q\}$-ring A is a $\{\delta_p, \delta_q\}$-*subring* of the $\{\delta_p, \delta_q\}$-ring B if A is a subring of B such that $\delta_p A \subset A$ and $\delta_q A \subset A$, respectively.

We describe next the basic examples of $\{\delta_p, \delta_q\}$-rings that will play crucial rôles in our paper. All throughout, we shall let R stand for $R := R_p := \widehat{\mathbb{Z}}_p^{ur}$, the completion of the maximum unramified extension of $\mathbb{Z}_p$, k for the residue field $k := R/pR$, and K for the fraction field $K := R[1/p]$. Furthermore, we will let $\mu(R)$ be the multiplicative group of roots of unity in R, and recall that the reduction mod p mapping

$$\mu(R) \to k^\times$$

defines an isomorphism whose inverse is the *Teichmüller lift*. Any element of the ring R can be represented uniquely as a series $\sum_{i=0}^\infty \zeta_i p^i$, where $\zeta_i \in \mu(R) \cup \{0\}$. There is a unique ring isomorphism

$$\phi : R \to R \tag{20}$$

that lifts the p-th power Frobenius isomorphism on k, and for $\zeta \in \mu(R)$, we have that $\phi(\zeta) = \zeta^p$.

The ring R is isomorphic to the Witt ring on the algebraic closure $\mathbb{F}_p^a$ of $\mathbb{F}_p$, and for each $s \geq 1$, the ring

$$R^{\phi^s} := \{x \in R \mid x^{\phi^s} = x\}$$

is isomorphic to the Witt ring on the field $\mathbb{F}_{p^s}$ with p^s elements. Notice that the ring $R^\phi = \mathbb{Z}_p$ is simply the ring of p-adic integers. As usual, we denote by $\mathbb{Z}_{(p)}$ the ring of all fractions $a/b \in \mathbb{Q}$, where $a \in \mathbb{Z}$ and $b \in \mathbb{Z} \setminus (p)$. We have the inclusions

$$\mathbb{Z}_{(p)} \subset \mathbb{Z}_p \subset R^{\phi^s} \subset R \, .$$

Example 2.3. The ring R carries a unique p-derivation $\delta_p : R \to R$ given by (see (17))

$$\delta_p x = \frac{\phi_p(x) - x^p}{p} \, .$$

The *constants of* δ_p are defined to be

$$R^{\delta_p} := \{x \in R \, : \, \delta_p x = 0\} \, ,$$

set that coincides with $\mu(R)\cup\{0\}$, the roots of unity in R together with 0. Notice that we have the trivial δ_p-derivation $\delta_q=0$ on R, and that the pair $(\delta_p,0)$ equips R with a $\{\delta_p,\delta_q\}$-ring structure.

We denote by $R[[q]]$ and $R[[q^{-1}]]$ the power series rings in q and q^{-1}, respectively, and we embed them into the rings

$$R((q))\hat{} = R[[q]][q^{-1}]\hat{} = \left\{ \sum_{n=-\infty}^{\infty} a_n q^n \mid \lim_{n\to-\infty} a_n = 0 \right\}$$

and

$$R((q^{-1}))\hat{} = R[[q^{-1}]][q]\hat{} = \left\{ \sum_{n=-\infty}^{\infty} a_n q^n \mid \lim_{n\to\infty} a_n = 0 \right\} ,$$

respectively.

The rings $R((q))\hat{}$ and $R((q^{-1}))\hat{}$ have unique structures of $\{\delta_p,\delta_q\}$-rings that extend that of R, such that

$$\delta_p q = 0\,,\ \delta_q q = q\,,\ \delta_p(q^{-1}) = 0\,,\ \delta_q(q^{-1}) = -q^{-1}\,.$$

The rings $R[[q]]$ and $R[[q^{-1}]]$ are $\{\delta_p,\delta_q\}$-subrings of $R((q))\hat{}$ and $R((q^{-1}))\hat{}$, respectively. Also the ring

$$R[q,q^{-1}]\hat{} = \left\{ \sum_{n=-\infty}^{\infty} a_n q^n \mid \lim_{n\to\pm\infty} a_n = 0 \right\}$$

is a $\{\delta_p,\delta_q\}$-subring of both $R((q))\hat{}$ and $R((q^{-1}))\hat{}$.

We describe these $\{\delta_p,\delta_q\}$-structures in further detail.

The automorphism (20) extends to a unique homomorphism $\phi: R[[q]]\to R[[q]]$ such that $\phi(q)=q^p$. Similarly, it extends to a unique homomorphism $\phi: R[[q^{-1}]]\to R[[q^{-1}]]$ such that $\phi(q^{-1})=q^{-p}$, to which for obvious reasons we give the same name. Then the expression

$$\delta_p F := \frac{F^{\phi}-F^p}{p}\,.$$

defines a p-derivation δ_p on both, $R[[q]]$ and $R[[q^{-1}]]$, respectively. On the other hand, the expression

$$\delta_q F := q\frac{dF}{dq} = q\partial_q F$$

defines a δ_p-derivation on $R((q))\hat{}$ and $R((q^{-1}))\hat{}$. Here,

$$\partial_q\left(\sum a_n q^n\right) = \frac{d}{dq}\left(\sum a_n q^n\right) := \sum n a_n q^n\,.$$

The operators δ_p and δ_q so defined provide the various rings above with their respective $\{\delta_p,\delta_q\}$-ring structures.

Observe that the R-algebra mapping

$$\begin{array}{ccc} R((q))\hat{} & \to & R((q^{-1}))\hat{} \\ q & \mapsto & q^{-1} \end{array}$$

is a ring isomorphism that fails to be a $\{\delta_p,\delta_q\}$-ring isomorphism. $\square$

In the sequel, we will repeatedly use the following "Dwork's Lemma," which we record here for convenience.

Lemma 2.4. *Let $v \in 1 + qK[[q]]$ be such that $v^{\phi}/v^p \in 1 + pqR[[q]]$. Then $v \in 1 + qR[[q]]$.*

Proof. For $v \in 1 + q\mathbb{Q}_p[[q]]$, this is proved in [19], p. 93. The general case follows by a similar argument. □

We will also need the following case of Hazewinkel's Functional Equation Lemma. This special version of this result implies Dwork's Lemma.

Lemma 2.5. [16] *Let $\mu_0 \in R^{\times}$, and $\mu_1, \dots, \mu_s \in R$. Consider two series $f, g \in K[[q]]$ such that $f \equiv \lambda q \mod q^2$ in $K[[q]]$, for some $\lambda \in R^{\times}$. Let $f^{-1} \in qK[[q]]$ be the compositional inverse of f. For*

$$\Lambda := \mu_0 + \frac{\mu_1}{p}\phi_p + \cdots + \frac{\mu_s}{p}\phi_p^s \in K[\phi_p] \, ,$$

assume that both, Λf and Λg, belong to $R[[q]]$. Then $f^{-1} \circ g \in R[[q]]$.

Closely related to these results is the following.

Lemma 2.6. *Let $\mu_0 \in R^{\times}$, $\mu_1, \dots, \mu_s \in R$, and $f \in K[[q]]$. If for*

$$\Lambda := \mu_0 + \mu_1\phi_p + \cdots + \mu_s\phi_p^s \in R[\phi_p]$$

we have that $\Lambda f \in qR[[q]]$, then $f \in qR[[q]]$. If on the other hand we have that $\Lambda f \in pqR[[q]]$, then $f \in pqR[[q]]$.

Proof. We may assume $\mu_0 = 1$. Set $\Lambda_1 := 1 - \Lambda \in \phi_p R[\phi_p]$, and note that $\Lambda_1 qR[[q]] \subset q^p R[[q]]$. Set $g = \Lambda f$. Then

$$f = g + \Lambda_1 g + \Lambda_1^2 g + \cdots \, ,$$

and the result follows. □

Formally speaking, the $\{\delta_p, \delta_q\}$-rings $R((q))\widehat{\ }$ and $R((q^{-1}))\widehat{\ }$, and their various $\{\delta_p, \delta_q\}$-subrings introduced above, can be viewed as rings of "Fourier series." These rings allow us to take limits of solutions to arithmetic partial differential equations in the "time" direction, as $t \to \pm i\infty$. On the other hand, the examples of $\{\delta_p, \delta_q\}$-rings of "Iwasawa series," which we discuss next, control limits "as time goes to 0."

Example 2.7. Let $R[[q-1]]$ be the completion of the polynomial ring $R[q]$ with respect to the ideal $(q-1)$. Hence, $R[[q-1]]$ can be identified with the power series ring $R[[\tau]]$ in an indeterminate τ, and $R[q]\widehat{\ }$ embeds into $R[[q-1]] = R[[\tau]]$ via the map $q \mapsto 1 + \tau$.

There is a unique $\{\delta_p, \delta_q\}$-ring structure on $R[[q-1]]$ that extends that of $R[q]\widehat{\ }$. Indeed, in terms of τ we have

$$\begin{array}{rcl} \delta_p \tau & = & \dfrac{(1+\tau)^p - 1 - \tau^p}{p} \, , \\ \delta_q \tau & = & 1 + \tau \, . \end{array}$$

Note that although $R[[q]]$ and $R[[q-1]]$ are isomorphic as rings, they are not isomorphic as $\{\delta_p, \delta_q\}$-rings. In fact, they are quite "different" in this latter context.

Similarly, let $R[[q^{-1}-1]]$ be the completion of $R[q^{-1}]$ with respect to the ideal $(q^{-1}-1)$. Then there is a natural embedding $R[q^{-1}]\widehat{\ } \subset R[[q^{-1}-1]]$, and a unique structure of $\{\delta_p, \delta_q\}$-ring on $R[[q^{-1}-1]]$ that extends that of $R[q^{-1}]\widehat{\ }$. □

From here on, we consider the ring A provided with a fixed structure of $\{\delta_p,\delta_q\}$-ring. For simplicity, we always assume that A is a p-adically complete Noetherian integral domain of characteristic zero, and we shall let L be its fraction field. Example 2.3 above illustrates the cases where A is equal to R, $R[[q]]$, $R((q))\widehat{\ }$, $R[[q^{-1}]]$, $R((q^{-1}))\widehat{\ }$, and $R[q,q^{-1}]\widehat{\ }$, respectively, while Example 2.7 illustrates the cases where A is equal to $R[[q-1]]$ and $R[[q^{-1}-1]]$, respectively. Whenever any of these rings is considered below, it will be given the $\{\delta_p,\delta_q\}$-ring structure defined in these two examples.

In Examples 2.8 and 2.9 below we explain some basic general constructions that can be performed using a fixed $\{\delta_p,\delta_q\}$-ring A.

Example 2.8. We will denote by $(A[\phi_p,\delta_q],+,\,\cdot\,)$ the ring generated by A and the symbols ϕ_p and δ_q subject to the relations

$$\begin{array}{rcl} \delta_q\cdot\phi_p & = & p\cdot\phi_p\cdot\delta_q\,, \\ [\delta_q,a]=\delta_q\cdot a-a\cdot\delta_q & = & \delta_q a\,, \\ \phi_p\cdot a & = & a^{\phi}\cdot\phi_p\,, \end{array}$$

for all $a\in A$. Let ξ_p, ξ_q be two variables, which we view as "duals to p and q," respectively. If $\mu=\mu(\xi_p,\xi_q)=\sum\mu_{ij}\xi_p^i\xi_q^j\in A[\xi_p,\xi_q]$ is a polynomial, then we define

$$\mu(\phi_p,\delta_q):=\sum\mu_{ij}\phi_p^i\delta_q^j\in A[\phi_p,\delta_q]\,.$$

The map

$$\begin{array}{rcl} A[\xi_p,\xi_q] & \to & A[\phi_p,\delta_q] \\ \mu(\xi_p,\xi_q) & \mapsto & \mu(\phi_p,\delta_q) \end{array}$$

is a left A-module isomorphism.

Given a $\mu(\xi_p,\xi_q)$ as above, we define the polynomial

$$\mu^{(p)}(\xi_p,\xi_q):=\mu(p\xi_p,\xi_q)\,.$$

We have the following useful formula:

$$\delta_q\mu(\phi_p,\delta_q)=\mu^{(p)}(\phi_p,\delta_q)\delta_q\,.$$

□

Example 2.9. Let y be an N-tuple of indeterminates over A, and let $y^{(i,j)}$ be an N-tuple of indeterminates over A parameterized by non-negative integers i, j, such that $y^{(0,0)}=y$. We set

$$A\{y\}:=A[y^{(i,j)}\mid_{i\geq 0,j\geq 0}]$$

for the polynomial ring in the indeterminates $y^{(i,j)}$. This ring has a natural $\{\delta_p,\delta_q\}$-structure that we now discuss.

For let $\phi_p : A\{y\}\to A\{y\}$ be the unique ring homomorphism extending $\phi_p : A\to A$, and satisfying the relation

$$\phi_p(y^{(i,j)})=(y^{(i,j)})^p+py^{(i+1,j)}\,.$$

Then, we may define a p-derivation $\delta_p : A\{y\}\to A\{y\}$ by the expression (see (17) above)

$$\delta_pF:=\frac{\phi_p(F)-F^p}{p}\,.$$

In particular, we have that $\delta_p y^{(i,j)}=y^{(i+1,j)}$.

We let $L\{y\}$ be $L\{y\} := L \otimes_A A\{y\}$, L the field of fractions of A. Notice that

$$L\{y\} = L[y^{(i,j)} \mid_{i\geq 0, j\geq 0}] = L[\phi_p^i y^{(0,j)} \mid_{i\geq 0, j\geq 0}]\,.$$

The endomorphism ϕ_p extends uniquely to an endomorphism ϕ_p of $L\{y\}$. Since the polynomials $\phi^i y^{(0,j)}$ are algebraically independent over L, there exists a unique derivation $\delta_q : L\{y\} \to L\{y\}$ that extends the derivation $\delta_q : A \to A$, and satisfies the relation

$$\delta_q(\phi_p^i y^{(0,j)}) = p^i \phi_p^i y^{(0,j+1)}\,.$$

We claim that $D := \delta_q \circ \phi_p - p \cdot \phi_p \circ \delta_q$ vanishes on $L\{y\}$.

Indeed, $D : L\{y\} \to L\{y\}$ is a derivation if we view $L\{y\}$ as an algebra over itself via ϕ_p. Thus, it suffices to observe that D vanishes on L and $\phi_p^i y^{(0,j)}$, and both of these are clear.

We now claim also that $\delta_q y^{(i,j)} \in A\{y\}$, and this will imply that δ_q induces a derivation of the ring $A\{y\}$. For, proceeding by induction on the index i, with the case $i = 0$ being clear, we assume it for i, and prove the desired statement for $i+1$. We have

$$\begin{array}{rcl} \delta_q y^{(i+1,j)} & = & \delta_q \left(\dfrac{\phi_p(y^{(i,j)}) - (y^{(i,j)})^p}{p} \right) \\ & = & \phi_p(\delta_q y^{(i,j)}) - (y^{(i,j)})^{p-1} \delta_q y^{(i,j)}\,, \end{array}$$

and this is clearly in $A\{y\}$ by the induction hypothesis.

The morphisms δ_p, δ_q endow $A\{y\}$ with a $\{\delta_p, \delta_q\}$-ring structure. Clearly, $y^{(i,j)} = \delta_p^i \delta_q^j y$, and so

$$A\{y\} = A[y, Dy, D^2 y, \ldots]\,,$$

where for any whole number n, $D^n y$ stands for the $(n+1)$-tuple with components $\delta_p^i \delta_q^{n-1} y$, $0 \leq i \leq n$. $\square$

Definition 2.10. Let $S^* = \{S^n\}_{n\geq 0}$ be a sequence of rings. Suppose we have ring homomorphisms $\varphi : S^n \to S^{n+1}$, p-derivations $\delta_p : S^n \to S^{n+1}$, and δ_p-derivations $\delta_q : S^n \to S^{n+1}$ such that $\delta_p \circ \varphi = \varphi \circ \delta_p$, and $\delta_q \circ \varphi = \varphi \circ \delta_q$. We then say that $(S^*, \varphi, \delta_p, \delta_q)$, or simply S^*, is a $\{\delta_p, \delta_q\}$-*prolongation sequence*. A *morphism* $u^* : S^* \to \tilde{S}^*$ of $\{\delta_p, \delta_q\}$-prolongation sequences is a sequence $u^n : S^n \to \tilde{S}^n$ of ring homomorphisms such that $\delta_p \circ u^n = u^{n+1} \circ \delta_p$, $\delta_q \circ u^n = u^{n+1} \circ \delta_q$, and $\varphi \circ u^n = u^{n+1} \circ \varphi$.

Let us consider the $\{\delta_p, \delta_q\}$-ring structure on the ring A. We obtain a natural $\{\delta_p, \delta_q\}$-prolongation sequence A^* by setting $A^n = A$ for all n, and taking the ring homomorphisms φ to be all equal to the identity. This leads to the natural concept of morphisms of $\{\delta_p, \delta_q\}$-prolongation sequences over A.

Definition 2.11. We say that a $\{\delta_p, \delta_q\}$-prolongation sequence S^* is a $\{\delta_p, \delta_q\}$-*prolongation sequence over* A if we have a morphism of prolongation sequences $A^* \to S^*$.

There is a natural notion of morphisms of $\{\delta_p, \delta_q\}$-prolongation sequences over A that we do not explicitly state. In the sequel, all $\{\delta_p, \delta_q\}$-prolongation sequences, and morphisms of such, will be prolongation sequences over A.

Example 2.12. Let $A\{y\}$ the ring discussed in Example 2.9. We consider the subrings

$$S^n := A[y, Dy, \ldots, D^n y]\,.$$

We view S^{n+1} as an S^n-algebra via the inclusion homomorphism, and observe that $\delta_p S^n \subset S^{n+1}$, and $\delta_q S^n \subset S^{n+1}$, respectively. Therefore, $S^* = \{S^n\}$ defines a $\{\delta_p, \delta_q\}$-prolongation sequence. We then obtain the p-adic completion prolongation sequence

$$A[y, Dy, \ldots, D^n y]\widehat{\ }\ ,$$

and the prolongation sequence

$$A[[Dy, \ldots, D^n y]]$$

of formal power series ring, with their corresponding $\{\delta_p, \delta_q\}$-structures. $\square$

Definition 2.13. Let X and Y be smooth schemes over the fixed $\{\delta_p, \delta_q\}$-ring A. By a $\{\delta_p, \delta_q\}$-*morphism of order* r we mean a rule $f : X \to Y$ that attaches to any $\{\delta_p, \delta_q\}$-prolongation sequence S^* of p-adically complete rings, a map of sets $X(S^0) \to Y(S^r)$ that is "functorial" in S^* in the obvious sense.

For any $\{\delta_p, \delta_q\}$-prolongation sequence S^*, the shifted sequence $S^*[i]$, $S[i]^n := S^{n+i}$, is a new $\{\delta_p, \delta_q\}$-prolongation sequence. Thus, any morphism $f : X \to Y$ of order r induces maps of sets $X(S^i) \to Y(S^{r+i})$ that are functorial in S^*. We can compose $\{\delta_p, \delta_q\}$-morphisms $f : X \to Y$, $g : Y \to Z$ of orders r and s, respectively, and get a $\{\delta_p, \delta_q\}$-morphisms $g \circ f : X \to Z$ of order $r + s$. There is a natural map from the set of $\{\delta_p, \delta_q\}$-morphisms $X \to Y$ of order r into the set of $\{\delta_p, \delta_q\}$-morphisms $X \to Y$ of order $r + 1$, induced by the maps $Y(S^r) \to Y(S^{r+1})$ arising from the S^r-algebra structure of S^{r+1}.

Recall that given a scheme X over the ring A, if B is an A-algebra we let $X(B)$ denote the set of all morphisms of A-schemes $\mathrm{Spec}\, B \to X$, and any such morphism is called a B-*point* of X. For instance, recall that if $X = \mathbb{A}^1 = \mathbb{G}_a = \mathrm{Spec}\, A[y]$, then $X(B)$ is simply the set B itself because a morphism $\mathrm{Spec}\, B \to \mathrm{Spec}\, A[y]$ is the same as a morphism $A[y] \to B$, and the latter is uniquely determined by the image of y in B. If on the other hand, $X = \mathbb{G}_m = \mathrm{Spec}\, A[y, y^{-1}] = \mathrm{Spec}\, A[x, y]/(xy - 1)$, then $X(B) = B^\times$ because $\mathrm{Hom}_A(A[y, y^{-1}], B) = B^\times$ via the map $f \mapsto f(y)$. Finally, if $X = \mathrm{Spec}\, A[x, y]/(f(x, y))$, then $X(B) = \{(a, b) \in B^2 :\ f(a, b) = 0\}$.

Definition 2.14. Let G and H be smooth group schemes over A. We say that $G \to H$ is a $\{\delta_p, \delta_q\}$-*homomorphism of order* r if it is a $\{\delta_p, \delta_q\}$-morphism of order r such that, for any prolongation sequence S^*, the maps $X(S^0) \to Y(S^r)$ are group homomorphisms. A $\{\delta_p, \delta_q\}$-*character* of order r of G is a $\{\delta_p, \delta_q\}$-homomorphism $G \to \mathbb{G}_a$ of order r, where $\mathbb{G}_a = \mathrm{Spec}\, A[y]$ is the additive group scheme over A. The group of $\{\delta_p, \delta_q\}$-characters of order r of G will be denoted by $\mathbf{X}^r_{pq}(G)$.

Note that the group $\mathbf{X}^r_{pq}(G)$ has a natural structure of $A[\phi_p, \delta_q]$-module. We shall view the $\{\delta_p, \delta_q\}$-characters of G as "linear partial differential operators" on it. Of course, they are highly "non-linear" in the affine coordinates around various points of G.

Let $\psi : G \to \mathbb{G}_a$ be a $\{\delta_p, \delta_q\}$-character of a commutative smooth group scheme G over A. For any $\{\delta_p, \delta_q\}$-ring B over A, ψ induces a $\mathbb{Z}$-linear map

$$\psi : G(B) \to \mathbb{G}_a(B) = B\,. \tag{21}$$

Definition 2.15. The *group of solutions* of ψ in B is the kernel of the map (21), that is to say, the group

$$\{u \in G(B) \mid \psi u = 0\}\,.$$

Given a subgroup Γ of $G(B)$, the *group of solutions* of ψ in Γ is will be the group

$$\{u \in \Gamma \mid \psi u = 0\}\,.$$

Example 2.16. Let us assume that $A = R$, and that G is a commutative smooth group scheme over R. We may consider the natural ring homomorphisms

$$\begin{array}{rcl} R[[q]] & \rightarrow & R \\ U(q) = \sum_{n\geq 0} a_n q^n & \mapsto & U(0) := \left(\sum_{n\geq 0} a_n q^n\right)_{|q=0} = a_0 \end{array}$$

and

$$\begin{array}{rcl} R[[q^{-1}]] & \rightarrow & R \\ U(q) = \sum_{n\leq 0} a_n q^n & \mapsto & U(\infty) := \left(\sum_{n\leq 0} a_n q^n\right)_{|q^{-1}=0} = a_0 \end{array}$$

respectively, and the corresponding induced group homomorphisms

$$\begin{array}{ccc} G(R[[q]]) & \rightarrow & G(R) \\ u & \mapsto & u(0) \end{array} \tag{22}$$

and

$$\begin{array}{ccc} G(R[[q^{-1}]]) & \rightarrow & G(R) \\ u & \mapsto & u(\infty) \end{array}\,. \tag{23}$$

If e denotes the identity, we have the natural diagram of groups

$$\begin{array}{ccccc} G(qR[[q]]) & \rightarrow & G(R[[q]]) & \rightarrow & G(R((q))\hat{\ }) \\ \uparrow & & \uparrow & & \uparrow \\ \{e\} & \rightarrow & G(R) & \rightarrow & G(R[q,q^{-1}]\hat{\ }) \\ \downarrow & & \downarrow & & \downarrow \\ G(q^{-1}R[[q^{-1}]]) & \rightarrow & G(R[[q^{-1}]]) & \rightarrow & G(R((q^{-1}))\hat{\ }) \end{array}\,, \tag{24}$$

where $G(qR[[q]])$ and $G(q^{-1}R[[q^{-1}]])$ are the kernels of the maps in (22) and (23), respectively.

We will think of the spectrum of the ring $A = R((q))\hat{\ }$ as the pq "plane." The p-axis is the "arithmetic" direction, while the q-axis is "geometric;" q will be viewed as the "exponential of $-2\pi i t$" where t is "time." The operators δ_p and δ_q are "vector fields" along these two directions.

Under these interpretations, the elements of $R((q))\hat{\ }$ play the role of "functions in the variables" p, q. By considering (infinitely many) negative powers of q, we allow the corresponding "function" to have an "essential singularity as time approaches $-i\infty$." The elements $u = \sum_{n\geq 0} a_n q^n$ of the power series ring $R[[q]]$ inside $R((q))\hat{\ }$ are viewed as the analogues of functions in two variables that, as time goes to infinity, approach a well defined limit function of the spatial variable, the function $a_0 = u(0)$. Here, the elements of the monoid $R^{\delta_p} = \{x \in R : \ \delta_p x = 0\}$ are to be thought of as the "constant functions."

Let ψ be a $\{\delta_p, \delta_q\}$-character of G. Considering the groups of solutions of ψ in the various groups of (24) above, we get a diagram

$$\begin{array}{ccccc} \mathcal{U}_1 & \rightarrow & \mathcal{U}_+ & \rightarrow & \mathcal{U}_\rightarrow \\ \uparrow & & \uparrow & & \uparrow \\ \{e\} & \rightarrow & \mathcal{U}_0 & \rightarrow & \mathcal{U}_\downarrow \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{U}_{-1} & \rightarrow & \mathcal{U}_- & \rightarrow & \mathcal{U}_\leftarrow \end{array}\,. \tag{25}$$

We will usually denote by $\mathcal{U}_*$ the groups in the above diagram. The elements of $\mathcal{U}_0$ will be referred to as *stationary solutions* of ψ, and are interpreted as solutions that "do not depend on time."

For $q_0 \in pR$, we may consider the ring homomorphism

$$\begin{array}{ccc} R[[q]] & \to & R \\ q & \mapsto & q_0 \end{array} .$$

This is a $\{\delta_p, \delta_q\}$-ring homomorphism for $q_0 = 0$, but not for $q_0 \neq 0$. The embedding $\mathrm{Spec}\, R \to \mathrm{Spec}\, R[[q]]$ will be viewed as a "curve" in the "pq-plane."

Corresponding to the homomorphism above, we may consider the specialization homomorphism

$$\begin{array}{ccc} G(R[[q]]) & \to & G(R) \\ u & \mapsto & u(q_0) \end{array} .$$

The "curve" associated to $q_0 \neq 0$ will be thought of as a curve along which we impose "boundary conditions." For if $u_0 \in G(R)$, we will consider the "boundary value problem" with that initial data, that is to say, the problem of finding a solution (possibly unique) $u \in \mathcal{U}_+$ such that $u(q_0) = u_0$. When $q_0 = 0$, $u(0) \in G(R)$ is thought of as the "limit of u as time goes to infinity," as mentioned earlier. This situation can be similarly stated for $R[[q^{-1}]]$ instead of $R[[q]]$.

We claim that the following hold:

$$\begin{array}{rclll} \psi(u(0)) & = & (\psi u)(0) & \text{for} & u \in G(R[[q]])\,, \\ \psi(u(\infty)) & = & (\psi u)(\infty) & \text{for} & u \in G(R[[q^{-1}]])\,. \end{array} \tag{26}$$

Indeed, in order to prove the first of these identities, it is enough to observe that, by the functorial definition of ψ, we have a commutative diagram

$$\begin{array}{ccc} G(R[[q]]) & \stackrel{\psi}{\to} & R[[q]] \\ \downarrow & & \downarrow \\ G(R) & \stackrel{\psi}{\to} & R \end{array} ,$$

where the vertical arrows are induced by the $\{\delta_p, \delta_q\}$-ring homomorphism

$$\begin{array}{ccc} R[[q]] & \to & R \\ q & \mapsto & 0 \end{array} .$$

The argument for the second of the identities is similar.

The identities in (26) imply that

$$\mathcal{U}_\pm = \mathcal{U}_0 \times \mathcal{U}_{\pm 1}, \tag{27}$$

where $\times$ denotes the "internal direct product."

We may unify this picture by considering the various groups of solutions $\mathcal{U}_*$ as subgroups of a single group. Indeed, let us consider the direct sum homomorphism

$$\psi \oplus \psi \,:\, \frac{G(R((q))\widehat{\ }) \oplus G(R((q^{-1}))\widehat{\ })}{G(R[q,q^{-1}]\widehat{\ })} \to \frac{R((q))\widehat{\ } \oplus R((q^{-1}))\widehat{\ }}{R[q,q^{-1}]\widehat{\ }}\,,$$

where the denominators in the domain and range are diagonally embedded into the numerators. We then set

$$\mathcal{U} := \ker(\psi \oplus \psi)\,,$$

and refer to this group as the *group of generalized solutions* of ψ. Its elements are the analogues of the generalized solutions of Sato.

The groups $\mathcal{U}_\leftarrow$ and $\mathcal{U}_\rightarrow$ naturally embed into $\mathcal{U}$ via the maps $x \mapsto (-x, 0)$ and $x \mapsto (0, x)$, respectively, and the restrictions of these two embeddings to $\mathcal{U}_\downarrow$ coincide.

Thus, all the groups $\mathcal{U}_*$ in the diagram (25) can be identified with subgroups of $\mathcal{U}$, and we have

$$\begin{array}{rcl} \mathcal{U}_\rightarrow \cap \mathcal{U}_\leftarrow & = & \mathcal{U}_\downarrow \, , \\ \mathcal{U}_+ \cap \mathcal{U}_- & = & \mathcal{U}_0 \, , \\ \mathcal{U}_1 \cap \mathcal{U}_{-1} & = & \{e\} \, . \end{array}$$

Note that, a priori, we do not have $\mathcal{U} = \mathcal{U}_\rightarrow \cdot \mathcal{U}_\leftarrow$.

Next we introduce a concept of *convolution* in our setting. Let $\mathbb{Z}\mu(R)$ be the group ring of the group $\mu(R)$. That is to say, $\mathbb{Z}\mu(R)$ is the set of all functions $f : \mu(R) \to \mathbb{Z}$ of finite support, equipped with pointwise addition, and multiplication $\star$ given by convolution

$$(f_1 \star f_2)(\zeta) := \sum_{\zeta_1 \zeta_2 = \zeta} f_1(\zeta_1) f_2(\zeta_2) \, .$$

For any integer $\kappa \in \mathbb{Z}$, we consider the ring homomorphism

$$\begin{array}{ccc} \mathbb{Z}\mu(R) & \stackrel{[\kappa]}{\to} & \mathbb{Z}\mu(R) \\ f & \mapsto & f^{[\kappa]} \end{array} \, , \tag{28}$$

where $f^{[\kappa]}(\zeta) := f(\zeta^\kappa)$. If $\mathbb{Z}[\mu(R)] \subset R$ is the subring generated by $\mu(R)$, then we have a natural surjective (but not injective) ring homomorphism

$$\begin{array}{ccc} \mathbb{Z}\mu(R) & \to & \mathbb{Z}[\mu(R)] \\ f & \mapsto & f^\sharp := \sum_\zeta f(\zeta)\zeta \end{array} \, .$$

Now let $R((q^{\pm 1}))\widehat{\ }$ be either $R((q))\widehat{\ }$ or $R((q^{-1}))\widehat{\ }$, respectively. For $\zeta \in \mu(R)$, we may consider the R-algebra isomorphism

$$\begin{array}{rcl} \sigma_\zeta : R((q^{\pm 1}))\widehat{\ } & \to & R((q^{\pm 1}))\widehat{\ } \\ u(q) & \mapsto & u(\zeta q) \end{array}$$

that induces a group isomorphism

$$\sigma_\zeta : G(R((q^{\pm 1}))\widehat{\ }) \to G(R((q^{\pm 1}))\widehat{\ }) \, .$$

Hence, the group $G(R((q^{\pm 1}))\widehat{\ })$ acquires a natural structure of $\mathbb{Z}\mu(R)$-module via convolution. Indeed, for any $f \in \mathbb{Z}\mu(R)$ and $u \in G(R((q^{\pm 1}))\widehat{\ })$, we have

$$f \star u := \sum_\zeta f(\zeta) \sigma_\zeta(u) \, ,$$

essentially a "superposition" of translates of u.

Note now that the homomorphism $\sigma_\zeta : R((q^{\pm 1}))\widehat{\ } \to R((q^{\pm 1}))\widehat{\ }$ is actually a $\{\delta_p, \delta_q\}$-ring homomorphism: the commutation of σ_ζ and δ_q is clear, while the commutation of σ_ζ and δ_p follows because

$$\phi_p(\sigma_\zeta(q^{\pm 1})) = \phi_p(\zeta^{\pm 1} q^{\pm 1}) = \zeta^{\pm \phi} q^{\pm p} = \zeta^{\pm p} q^{\pm p} = \sigma_\zeta(q^{\pm p}) = \sigma_\zeta(\phi_p(q^{\pm})) \, .$$

If ψ is a $\{\delta_p, \delta_q\}$-character of G, by the functorial definition of $\{\delta_p, \delta_q\}$-characters we obtain a commutative diagram

$$\begin{array}{ccc} G(R((q^{\pm 1}))\widehat{\ }) & \stackrel{\psi}{\to} & R((q^{\pm 1}))\widehat{\ } \\ \sigma_\zeta \downarrow & & \downarrow \sigma_\zeta \\ G(R((q^{\pm 1}))\widehat{\ }) & \stackrel{\psi}{\to} & R((q^{\pm 1}))\widehat{\ } \end{array} \, .$$

Hence, for any $f \in \mathbb{Z}\mu(R)$ and $u \in G(R((q^{\pm 1}))\widehat{\ })$, we have

$$\psi(f \star u) = f \star \psi(u) \, .$$

That is to say, ψ is $\mathbb{Z}\mu(R)$- module homomorphism. In particular, the groups $\mathcal{U}_{\pm 1}$ of solutions of ψ in $G(R((q^{\pm 1}))\hat{})$ are $\mathbb{Z}\mu(R)$- submodules of $G(R((q^{\pm 1}))\hat{})$ respectively. Morally speaking, this says that a superposition of translates of a solution is again a solution.

Let us additionally assume that G has relative dimension one over R. Let T be an étale coordinate on a neighborhood of the zero section in G such the zero section is given scheme theoretically by $T=0$. Then the ring of functions on the completion of G along the zero section is isomorphic to a power series ring $R[[T]]$. We fix such an isomorphism. Then we have an induced group isomorphism

$$\iota : q^{\pm 1}R[[q^{\pm 1}]] \rightarrow G(q^{\pm 1}R[[q^{\pm 1}]])\, , \tag{29}$$

where $q^{\pm 1}R[[q^{\pm 1}]]$ is a group relative to a formal group law $\mathcal{F}(T_1,T_2)\in R[[T_1,T_2]]$ attached to G. We recall that the series $[p^n](T)\in R[[T]]$, defined by multiplication by p^n in the formal group $\mathcal{F}$, belongs to the ideal $(p,T)^n$. We equip $G(q^{\pm 1}R[[q^{\pm 1}]])$ with the topology induced via ι from the $(p,q^{\pm 1})$-adic topology on $R[[q^{\pm 1}]]$. Since $G(q^{\pm 1}R[[q^{\pm 1}]])$ is complete in this topology, the $\mathbb{Z}\mu(R)$-module structure of $G(q^{\pm 1}R[[q^{\pm 1}]])$ extends uniquely to a $\mathbb{Z}\mu(R)\hat{}$-module structure on $G(q^{\pm 1}R[[q^{\pm 1}]])$ in which multiplication by scalars is continuous. Here, $\mathbb{Z}\mu(R)\hat{}$ is, of course, the p-adic completion of $\mathbb{Z}\mu(R)$. We observe that the ring homomorphism

$$\begin{array}{ccc} \mathbb{Z}\mu(R)\hat{} & \stackrel{\sharp}{\rightarrow} & R \\ f & \mapsto & f^{\sharp} := \sum_{\zeta} f(\zeta)\zeta \end{array} \tag{30}$$

is surjective (but not injective). Clearly, the groups $\mathcal{U}_{\pm 1}$ of solutions of ψ are $\mathbb{Z}\mu(R)\hat{}$- submodules of $\mathcal{U}_{\pm}$, respectively.

Also, we note that a Mittag-Leffler argument shows that for any κ coprime to p the map $[\kappa] : \mathbb{Z}\mu(R)\hat{} \rightarrow \mathbb{Z}\mu(R)\hat{}$ is surjective.

We end our discussion here by introducing boundary value operators at $q^{\pm 1}=0$. We start by considering, for any $0\neq \kappa \in \mathbb{Z}$, operators

$$\begin{array}{ccc} \Gamma_{\kappa} : R[[q^{\pm 1}]] & \rightarrow & R \\ u=\sum a_n q^n & \mapsto & \Gamma_{\kappa} u = a_{\kappa} \end{array} .$$

We fix also a δ_q-character ψ_q of G. In the applications, for $G=\mathbb{G}_a$, ψ_q will be the identity; and for G either $\mathbb{G}_m$ or an elliptic curve E over R, ψ_q will be the "Kolchin logarithmic derivative." If we fix a collection of non-zero integers $\mathcal{K}_{\pm}\subset \mathbb{Z}_{\pm}$, and set $\rho_{\pm} := \sharp \mathcal{K}_{\pm}$, then we may consider the *boundary value operator at* $q^{\pm 1}=0$,

$$\begin{array}{ccc} G(R[[q^{\pm 1}]]) & \stackrel{B^0_{\pm}}{\rightarrow} & R^{\rho_{\pm}} \\ B^0_{\pm} u & = & (B^0_{\kappa} u)_{\kappa \in \mathcal{K}_{\pm}} \end{array} , \tag{31}$$

where $B^0_{\kappa}u := \Gamma_{\kappa}\psi_q u$. Here we view $R^{\rho_{\pm}}$ as a direct product $R^{\rho_{\pm}} = \prod_{\kappa\in\mathcal{K}_{\pm}} R_{\kappa}$ of copies R_{κ} of R indexed by $\mathcal{K}_{\pm}$.

It is easy to check that $B^0_{\pm}$ is a $\mathbb{Z}\mu(R)$-module homomorphism provided that $R^{\rho_{\pm}}$ be viewed as a $\mathbb{Z}\mu(R)$-module with the following structure: for each $\kappa \in \mathcal{K}_{\pm}$, the copy of R indexed by κ in $R^{\rho_{\pm}}$ is a $\mathbb{Z}\mu(R)$-module via the ring homomorphism

$$\mathbb{Z}\mu(R) \stackrel{[\kappa]}{\rightarrow} \mathbb{Z}\mu(R) \stackrel{\sharp}{\rightarrow} R\, . \tag{32}$$

In other words, for $f\in \mathbb{Z}\mu(R)$ and $(r_{\kappa})_{\kappa}\in R^{\rho_{\pm}}$, we have that

$$f\cdot (r_{\kappa})_{\kappa} := ((f^{[\kappa]})^{\sharp} r_{\kappa})_{\kappa}.$$

The restriction of $B^0_{\pm}$ to $G(q^{\pm 1}R[[q^{\pm 1}]])$,

$$B^0_{\pm} : G(q^{\pm 1}R[[q^{\pm 1}]]) \quad \to \quad R^{\rho_{\pm}}\,, \tag{33}$$

is a $\mathbb{Z}\mu(R)\widehat{\ }$-module homomorphism where $R^{\rho_\pm}$ is viewed as a $\mathbb{Z}\mu(R)\widehat{\ }$-module with the following structure: for each $\kappa \in \mathcal{K}_{\pm}$, the copy of R indexed by κ in $R^{\rho_\pm}$ is a $\mathbb{Z}\mu(R)\widehat{\ }$-module via the ring homomorphism

$$\mathbb{Z}\mu(R)\widehat{\ } \stackrel{[\kappa]}{\to} \mathbb{Z}\mu(R)\widehat{\ } \stackrel{\sharp}{\to} R\,. \tag{34}$$

It our applications, it will sometimes be the case that the $\mathbb{Z}\mu(R)\widehat{\ }$-module structure of $\mathcal{U}_{\pm 1}$ induces a certain structure of R-module, and the restriction of $B^0_{\pm}$ to $\mathcal{U}_{\pm 1}$ becomes an R-module homomorphism.

Remark 2.17. The discussion in the previous Example applies to the case where G is one of the following group schemes over $A = R$:

(1) $G = \mathbb{G}_a := \mathrm{Spec}\, R[y]$, the additive group.
(2) $G = \mathbb{G}_m := \mathrm{Spec}\, R[y, y^{-1}]$, the multiplicative group.
(3) $G = E$, an elliptic curve over R.

The discussion, however, does not apply to the following case, which will also interest us later:

(4) $G = E$, an elliptic curve over $A = R((q))\widehat{\ }$ that does not descend to R, for instance, the Tate curve.

In this latter case, we will need new definitions for some of the groups (25) (cf. the discussion in our subsection on the Tate curve).

3. Partial differential jet spaces of schemes

In this section we introduce arithmetic-geometric partial differential jet spaces of schemes. We do this by analogy with the ordinary differential ones in geometry [3] and arithmetic [5, 7, 9], respectively. We also record some of their general properties (that can be proven in a manner similar to the corresponding proofs in the ordinary case [9], proofs that, therefore, will be omitted here.) We recall that the $\{\delta_p, \delta_q\}$-ring A we have fixed is a p-adically complete Noetherian integral domain of characteristic zero.

For any scheme X of finite type over A, we define a projective system of p-adic formal schemes

$$\cdots \to J^r_{pq}(X) \to J^{r-1}_{pq}(X) \to \cdots \to J^1_{pq}(X) \to J^0_{pq}(X) = \hat{X}\,, \tag{35}$$

called the $\{\delta_p, \delta_q\}$-*jet spaces* of X.

Let us assume first that X is affine, that is to say, $X = \mathrm{Spec}\, A[y]/I$, where y is a tuple of indeterminates. We then set

$$J^n_{pq}(X) := Spf\ A[y, Dy, \ldots, D^n y]\widehat{\ }/(I, DI, \ldots, D^n I)\,.$$

The sequence of rings $\{\mathcal{O}(J^n_{pq}(X))\}_{n\geq 0}$ has a natural structure of $\{\delta_p, \delta_q\}$-prolongation sequence $\mathcal{O}(J^*_{pq}(X))$, and the latter has the following universality property that can be easily checked.

Proposition 3.1. *For any $\{\delta_p, \delta_q\}$-prolongation sequence S^* that consists of p-adically complete rings, and for any homomorphism $u : \mathcal{O}(X) \to S^0$ over A, there is a unique morphism of $\{\delta_p, \delta_q\}$-prolongation sequences*

$$u_* = (u_n) : \mathcal{O}(J^*_{pq}(X)) \to S^*\,,$$

with $u_0 = u$.

Proof. Similar to [9], Proposition 3.3. $\square$

As a consequence we get that, for affine X, the construction $X \mapsto J^r(X)$ is compatible with localization in the following sense:

Corollary 3.2. *If* $X = \operatorname{Spec} B$ *and* $U = \operatorname{Spec} B_f$, $f \in B$, *then*

$$\mathcal{O}(J^r_{pq}(U)) = (\mathcal{O}(J^r_{pq}(X))_f)\widehat{\ }.$$

Equivalently,

$$J^r_{pq}(U) = J^r_{pq}(X) \times_{\hat{X}} \hat{U}\,.$$

Proof. Similar to [9], Corollary 3.4. $\square$

Consequently, for X that is not necessarily affine, we can define a formal scheme $J^r_{pq}(X)$ by gluing the various schemes $J^r_{pq}(U_i)$ for $\{U_i\}$ an affine Zariski open covering of X. The resulting formal scheme will have a corresponding universality property, whose complete formulation and verification we leave to the reader.

Remark 3.3. If $\mathbb{G}_a = \operatorname{Spec} A[y]$ is the additive group scheme over A, then

$$J^n_{pq}(\mathbb{G}_a) = Spf\ A[y, Dy, \ldots, D^n y]\widehat{\ }.$$

If $\mathbb{G}_m = \operatorname{Spec} A[y, y^{-1}]$ is the multiplicative group scheme over A, then

$$J^n_{pq}(\mathbb{G}_m) = Spf\ A[y, y^{-1}, Dy, \ldots, D^n y]\widehat{\ }.$$

Remark 3.4. By the universality property of jet spaces, we have that

$$J^n_{pq}(X \times Y) \simeq J^n_{pq}(X) \times J^n_{pq}(Y)$$

where the product on the left hand side is taken in the category of schemes of finite type over A, and the product on the right hand side is taken in the category of formal schemes over A. In the same vein, if X is a group scheme of finite type over A, then (35) is a projective system of groups in the category of p-adic formal schemes over A.

Remark 3.5. By the universality property of jet spaces, the set of order r $\{\delta_p, \delta_q\}$-morphisms $X \to Y$ between two schemes of finite type over A naturally identifies with the set of morphisms over A of formal schemes $J^r_{pq}(X) \to J^0_{pq}(Y) = \hat{Y}$. In particular, the set $\mathcal{O}^r_{pq}(X)$ of all order r $\{\delta_p, \delta_q\}$-morphisms $X \to \hat{\mathbb{A}}^1$ identifies with the ring of global functions $\mathcal{O}(J^r_{pq}(X))$. If G is a group scheme of finite type over A, then the group $\mathbf{X}^r_{pq}(G)$ of order r $\{\delta_p, \delta_q\}$-characters $G \to \mathbb{G}_a$ identifies with the group of homomorphisms $J^r_{pq}(G) \to \hat{\mathbb{G}}_a$, and thus, it identifies with an A-submodule of $\mathcal{O}(J^r_{pq}(G))$. Let

$$\mathcal{O}^\infty_{pq}(X) := \lim_{\to} \mathcal{O}^r_{pq}(X)$$

be the $\{\delta_p, \delta_q\}$-ring of all $\{\delta_p, \delta_q\}$-morphisms $X \to \mathbb{A}^1$, and let

$$\mathbf{X}^\infty_{pq}(X) := \lim_{\to} \mathbf{X}^r_{pq}(X)$$

be the group of $\{\delta_p, \delta_q\}$-characters $G \to \mathbb{G}_a$. Then $\mathcal{O}^\infty_{pq}(X)$ has a natural structure of $A[\phi_p, \delta_q]$-module, and $\mathbf{X}^\infty_{pq}(X)$ is an $A[\phi_p, \delta_q]$-submodule.

Proposition 3.6. *Let X be a smooth affine scheme over A, and let $u : A[y] \to \mathcal{O}(X)$ be an étale morphism, where y is a d-tuple of indeterminates. Let $y^{(i,j)}$ be d-tuples of indeterminates, where $i, j \geq 0$. Then the natural morphism*

$$\mathcal{O}(\hat{X})[y^{(i,j)} \mid_{1 \leq i+j \leq n}]\widehat{\ } \to \mathcal{O}(J^n_{pq}(X))$$

that sends $y^{(i,j)}$ into $\delta_p^i \delta_q^j(u(y))$ is an isomorphism. In particular, we have an isomorphism of formal schemes over A

$$J^n_{pq}(X) \simeq \hat{X} \times \hat{\mathbb{A}}^{\frac{n(n+3)}{2}d}.$$

Proof. The argument is similar to that used in Proposition 3.13 of [9]. □

Corollary 3.7. *If $Y \to X$ is an étale morphism of smooth schemes over A then*

$$J^n_{pq}(Y) \simeq J^n_{pq}(X) \times_{\hat{X}} \hat{Y}.$$

Remark 3.8. The jet spaces with respect to a derivation [2] that arises in the Ritt-Kolchin theory [20] [26] are "covered" by of our $\{\delta_p, \delta_q\}$-jet spaces here. The same holds for the p-jet spaces (with respect to a p-derivation) constructed in [5]. More precisely, if we for affine X/A we set

$$\begin{array}{rcl} J^n_q(X) & := & \mathrm{Spec}\, A[y, \delta_q y, \ldots, \delta_q^n y]/(I, \delta_q I, \ldots, \delta_q^n I)\,, \\ J^n_p(X) & := & Spf\ A[y, \delta_p y, \ldots, \delta_p^n y]\widehat{\ }/(I, \delta_p I, \ldots, \delta_p^n I)\,, \end{array}$$

then we have natural morphisms

$$\begin{array}{rcl} J^n_{pq}(X) & \to & J^n_q(X)\widehat{\ }, \\ J^n_{pq}(X) & \to & J^n_p(X)\,. \end{array}$$

The same holds then for any scheme X, not necessarily affine. The elements of $\mathcal{O}^r_q(X) := \mathcal{O}(J^r_q(X))$ identify with the δ_q-morphisms $X \to \mathbb{A}^1$; the elements of $\mathcal{O}^r_p(X) := \mathcal{O}(J^r_p(X))$ identify with the δ_p-morphisms $X \to \mathbb{A}^1$. The homomorphisms $J^r_q(G) \to \hat{\mathbb{G}}_a$ identify with the δ_q-characters $G \to \mathbb{G}_a$; the homomorphisms $J^r_p(G) \to \hat{\mathbb{G}}_a$ identify with the δ_p-characters $G \to \mathbb{G}_a$.

Remark 3.9. The functors $X \mapsto J^r_{pq}(X)$ from the category $\mathcal{C}$ of A-schemes of finite type to the category $\hat{\mathcal{C}}$ of p-adic formal schemes naturally extends to a functor from $\mathcal{B}$ to $\hat{\mathcal{C}}$, where $\mathcal{B}$ is the category whose objects as the same as those in $\mathcal{C}$, hence $Ob\,\mathcal{C} = Ob\,\mathcal{B}$, and whose morphisms are defined by

$$\mathrm{Hom}_{\mathcal{B}}(X, Y) := \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{X}, \hat{Y})$$

for all $X, Y \in Ob\,\mathcal{B}$.

4. Partial differential jet spaces of formal groups

In this section we attach to any formal group law $\mathcal{F}$ in one variable, certain formal groups in several variables that should be thought of as arithmetic-geometric partial differential jets of $\mathcal{F}$. We use the interplay between these and the partial differential jet spaces of schemes introduced in the previous section to prove that the module of $\{\delta_p, \delta_q\}$-characters is finitely generated. We also define the notion of Picard-Fuchs symbol of a $\{\delta_p, \delta_q\}$-character, which will play a key rôle later.

Let A be our fixed $\{\delta_p, \delta_q\}$-ring and y a d-tuple of variables. For any $G \in A[[y, Dy, \ldots, D^n y]]$, we let $G_{|y=0} \in A[[Dy, \ldots, D^n y]]$ denote the series obtained from G by setting $y = 0$, while keeping $\delta_p^i \delta_q^j y$ unchanged for $i + j \geq 1$. We recall that L stands for the field of fractions of A.

Lemma 4.1. *For $a, b \in \mathbb{Z}_+$, we set $B := A[Dy, \dots, D^{a+b}y]\widehat{\ }$. Then:*

1) *If $G \in A[[y]][\delta_q y, \dots, \delta_q^b y]$, then $(\phi^a G)_{|y=0} \in B$.*
2) *If $F \in A[[y]]$, then $(\delta_p^a \delta_q^b F)_{|y=0} \in B$.*
3) *In the case where y is a single variable, if $F = \sum_{n\geq 1} c_n y^n \in yL[[y]]$ and $nc_n \in A$, then $(\phi_p^a \delta_q^b F)_{|y=0} \in B$ and $(\phi_p^a F)_{|y=0} \in pB$.*

Proof. For the proof assertion 1), we may assume that $G = a(y)\cdot(\delta_q y)^{i_1} \dots (\delta_q^b y)^{i_b}$, $a(y) \in A[[y]]$. We have that

$$y^{\phi^a} - y^{p^a} = p\Phi_a\,, \quad \Phi_a \in \mathbb{Z}[y, \delta_p y, \dots, \delta_p^a y]\,, \tag{36}$$

and so we get $(\phi_p^a G)_{|y=0} = (\phi_p^a a(y))_{|y=0} \cdot [(\phi_p^a \delta_q y)^{i_1} \dots (\phi_p^a \delta_q^b y)^{i_b}]_{|y=0}$.

It is clear that $[(\phi_p^a \delta_q y)^{i_1} \dots (\phi_p^a \delta_q^b y)^{i_b}]_{|y=0} \in \mathbb{Z}[Dy, \dots, D^{a+b}y]$. On the other hand, if $a(y) = \sum_{n\geq 0} a_n y^n$, by (36) we then have that

$$\begin{array}{rcl} (\phi_p^a a(y))_{|y=0} & = & (\sum_{n\geq 0} a_n^{\phi^a}(y^{p^a} + p\Phi_a)^n)_{|y=0} \\ & = & \sum_{n\geq 0} a_n^{\phi^a} p^n ((\Phi_a)_{|y=0})^n\,, \end{array}$$

which is an element of $A[\delta_p y, \dots, \delta_p^a y]\widehat{\ }$.

For the proof of the second part, we first use induction to check that $\delta_q^b F \in A[[y]][\delta_q y, \dots, \delta_q^b y]$. By the part of the Lemma already proven, we conclude that $(\phi_p^a \delta_q^b F)_{|y=0} \in A[Dy, \dots, D^{a+b}y]\widehat{\ }$. Assertion 2) then follows because $p^a \delta_p^a \delta_q^b F$ belongs to the ring generated by $\phi_p^i \delta_q^b F$ with $i \leq a$.

For assertion 3), we first use induction to check that

$$\delta_q^b F = \sum_{n\geq 1} (\delta_q^b c_n) y^n + G_b, \quad G_b \in A[[y]][\delta_q y, \dots, \delta_q^b y]\,. \tag{37}$$

Indeed, this is true for $b = 0$, with $G_0 = 0$, and if true for some b, then

$$\delta_q^{b+1} F = \sum_{n\geq 1} (\delta_q^{b+1} c_n) y^n + [\sum_{n\geq 1} (\delta_q^b c_n) n y^{n-1}] \delta_q y + \delta_q G_b\,,$$

by the induction hypothesis and the fact that $nc_n \in A$. By (37), we derive that

$$(\phi_p^a \delta_q^b F)_{|y=0} = [\sum_{n\geq 1} (\phi_p^a \delta_q^b c_n)(y^{\phi^a})^n]_{|y=0} + (\phi_p^a G_b)_{|y=0}\,.$$

By the second part of the Lemma proven above, $(\phi_p^a G_b)_{|y=0} \in A[\delta_p^i \delta_q^j y \mid_{1\leq i+j\leq a+b}]\widehat{\ }$. On the other hand

$$\begin{array}{rcl} [\sum_{n\geq 1} (\phi_p^a \delta_q^b c_n)(y^{\phi_p^a})^n]_{|y=0} & = & [\sum_{n\geq 1} (\phi_p^a \delta_q^b c_n)(y^{p^a} + p\Phi_a)^n]_{|y=0} \\ & = & \sum_{n\geq 1} (\phi_p^a \delta_q^b c_n) p^n ((\Phi_a)_{|y=0})^n\,, \end{array}$$

which belongs to $pA[Dy, \dots, D^{a+b}y]\widehat{\ }$ because $c_n p^n = nc_n(p^n/n) \in pA$, and it defines a sequence that converge to 0 as $n \to \infty$. $\square$

Let now T be one variable, (T_1, T_2) a pair of "copies" of T, and $\mathcal{F} := \mathcal{F}(T_1, T_2) \in A[[T_1, T_2]]$ be a formal group law in the variable T. Then, as in [9], p. 124, the tuple

$$(\mathcal{F}, D\mathcal{F}, \dots, D^n \mathcal{F}) \in A[[T_1, T_2, DT_1, DT_2, \dots, D^n T_1, D^n T_2]]^{\frac{(n+1)(n+2)}{2}} \tag{38}$$

is a formal group law over A in the variables $T, DT, \dots, D^n T$. Consequently, the tuple

$$(D\mathcal{F}_{|T_1=T_2=0}, \dots, D^n \mathcal{F}_{|T_1=T_2=0}) \in A[[DT_1, DT_2, \dots, D^n T_1, D^n T_2]]^{\frac{n(n+3)}{2}} \tag{39}$$

is a formal group law over A in the variables $DT, \dots, D^nT$. By the second part of Lemma 4.1, this series (39) belong to $A[DT_1, DT_2, \dots, D^nT_1, D^nT_2]\widehat{\ }$, so they define a structure of group objects in the category of formal schemes over A,

$$(\hat{\mathbb{A}}^{\frac{n(n+3)}{2}}, [+]). \tag{40}$$

Let $l = l(T) \in L[[T]]$ be the logarithm of $\mathcal{F}$ (cf. [28]), and let $a + b \leq n$. By assertion 3) of Lemma 4.1, we have that

$$L^{[a,b]} := p^{\epsilon(b)}(\phi_p^a \delta_q^b l)_{|T=0} \in A[DT, \dots, D^nT]\widehat{\ },$$

where $\epsilon(b)$ is either 0 or -1 if $b > 0$ or $b = 0$, respectively. So $L^{[a,b]}$ defines a morphism of formal schemes

$$L^{[a,b]} : \hat{\mathbb{A}}^{\frac{n(n+3)}{2}} \to \hat{\mathbb{A}}^1 .$$

As in [9], p. 125, the morphism $L^{[a,b]}$ above is actually a homomorphism

$$L^{[a,b]} : (\hat{\mathbb{A}}^{\frac{n(n+3)}{2}}, [+]) \to \hat{\mathbb{G}}_a = (\hat{\mathbb{A}}^1, +) \tag{41}$$

of group objects in the category of formal schemes.

We let G be $\mathbb{G}_a$, $\mathbb{G}_m$, or an elliptic curve (defined by a Weierstrass equation) over A, and let $\mathcal{F}$ be the formal group law naturally attached to G. In the case $G = \mathbb{G}_a = \mathrm{Spec}\, A[y]$, we let $T = y$. In the case $G = \mathbb{G}_m = \mathrm{Spec}\, A[y, y^{-1}]$, we let $T = y - 1$. And if $G = E$, we let T be an étale coordinate in a neighborhood U of the zero section 0 such that 0 is the zero scheme of T in U. The group

$$\ker(J^n_{pq}(G) \to J^0_{pq}(G) = \hat{G})$$

is isomorphic to the group (40).

Let $e(T) \in L[[T]]$ be the exponential of the formal group law $\mathcal{F}$ (that is to say, the compositional inverse of $l(T) \in L[[T]]$). We have $e(pT) \in pTA[T]\widehat{\ }$. We may consider the map

$$\begin{array}{ccc} A[[T]][DT, \dots, D^rT]\widehat{\ } & \stackrel{\circ e(pT)}{\longrightarrow} & A[T, DT, \dots, D^rT]\widehat{\ } \\ G \mapsto G \circ e(pT) & := & G(\delta_p^i \delta_q^j (e(pT)) \mid_{0 \leq i+j \leq r}) \end{array} . \tag{42}$$

On the other hand, by Proposition 3.6 we may consider the natural map

$$\mathcal{O}(J^r_{pq}(G)) \to A[[T]][DT, \dots, D^rT]\widehat{\ },$$

which by the said Proposition is injective with torsion free cokernel. We shall view this map as an inclusion.(Note that this is the case, more generally, when G is replaced by a smooth scheme over A and T is an étale coordinate.)

Lemma 4.2. *For any $\{\delta_p, \delta_q\}$-character $\psi \in \mathcal{O}(J^r_{pq}(G))$, the series $\psi \circ e(pT)$ is in the A-linear span of $\{\phi_p^i \delta_q^j T \mid_{0 \leq i+j \leq r}\}$.*

Proof. Clearly $F := \psi \circ e(pT)$ is *additive*, that is to say,

$$F(T_1 + T_2, \dots, D^r(T_1 + T_2)) = F(T_1, \dots, D^rT_1) + F(T_2, \dots, D^rT_2) .$$

But the only additive elements in

$$A[1/p][[T, DT, \dots, D^rT]] = A[1/p][[\phi_p^i \delta_q^j T \mid_{0 \leq i+j \leq r}]]$$

are those in the $A[1/p]$-linear span of $\{\phi_p^i \delta_q^j T \mid_{0 \leq i+j \leq r}\}$. We are thus left to show that if

$$\sum_{ij} a_{ij} \phi_p^i \delta_q^j T \in pA[T, DT, \dots, D^rT]\widehat{\ }$$

for $a_{ij} \in A$, then $a_{ij} \in pA$ for all i, j. Let $\bar{a}_{ij} \in A/pA$ be the images of a_{ij}. Then we have

$$\sum \bar{a}_{ij}(\delta_q^j T)^{p^i} = 0 \in (A/pA)[T, \delta_q T, \ldots, \delta_q^r T]\,,$$

which clearly implies $\bar{a}_{ij} = 0$ for all i, j. □

Corollary 4.3. *The A-module $\mathbf{X}^r_{pq}(G)$ of $\{\delta_p, \delta_q\}$-characters or order r is finitely generated of rank at most*

$$\frac{(r+1)(r+2)}{2}\,.$$

Proof. By Lemma 4.2, we have that $\mathbf{X}^r_{pq}(G)$ embeds into a finitely generated A-module of rank at most $\frac{(r+1)(r+2)}{2}$. The result follows because A is Noetherian. □

In light of Lemma 4.2, if ψ is a $\{\delta_p, \delta_q\}$-character of G then, as an element of the ring $A[[T]]DT, \ldots, D^r T]\widehat{\ }$, ψ can be identified with the series

$$\psi = \frac{1}{p}\sigma(\phi_p, \delta_q) l(T)\,, \tag{43}$$

where $\sigma = \sigma(\xi_p, \xi_q) \in A[\xi_p, \xi_q]$ is a polynomial.

Definition 4.4. The polynomial σ is the *Picard-Fuchs symbol* of ψ with respect to T.

Cf. [5] for comments on the terminology. In a more general context, we will later define what we call the *Fréchet symbol*, and will explain the relation between these two symbol notions.

The following Lemmas will also be needed later.

Lemma 4.5.

$$\left(L^{[a,b]} \circ e(pT)\right)_{|T=0} = \left(p^{1+\epsilon(b)}\phi_p^a \delta_q^b T\right)_{|T=0}\,.$$

Proof. We have

$$\begin{array}{rcl} \left(L^{[a,b]} \circ e(pT)\right)_{|T=0} & = & p^{\epsilon(b)}(\phi_p^a\delta_q^b l)(0, \delta_p(e(pT)), \delta_q(e(pT)), \ldots)_{|T=0} \\ & = & p^{\epsilon(b)}(\phi_p^a\delta_q^b l)(e(pT), \delta_p(e(pT)), \delta_q(e(pT)), \ldots)_{|T=0} \\ & = & (p^{\epsilon(b)}\phi_p^a\delta_q^b(l(e(pT))))_{|T=0} \\ & = & \left(p^{1+\epsilon(b)}\phi_p^a\delta_q^b T\right)_{|T=0}\,. \end{array}$$

□

Lemma 4.6. *The family $\{L^{[a,b]}\,|_{1\le a+b\le r}\}$ is A-linearly independent.*

Proof. By Lemma 4.5, it is enough to check that the family

$$\{(\phi_p^a\delta_q^b T)\,|_{T=0, 1\le a+b\le r}\}$$

is A-linearly independent. Let us assume that

$$\sum_{a+b\ge 1} \lambda_{ab}(\phi_p^a\delta_q^b T)_{|T=0} = 0\,.$$

This implies that

$$\sum_{a+b\ge 1} \lambda_{ab}\phi_p^a\delta_q^b T \in TA[T, DT, D^2T, \ldots] \subset TL[\phi_p^i\delta_q^j T\,|_{i\ge 0, j\ge 0}]\,,$$

which clearly implies $\lambda_{ab} = 0$ for all a, b. □

5. Fréchet derivatives and symbols

We now develop arithmetic-geometric analogues of some classical constructions in the calculus of variations, including Fréchet derivatives and Euler-Lagrange equations. We use the Fréchet derivatives to define Fréchet symbols of $\{\delta_p,\delta_q\}$-characters, and we relate Fréchet symbols to the previously defined Picard-Fuchs symbols.

5.1. **Fréchet derivative.** We recall that for a smooth scheme X over A, we denote by

$$T(X) := \mathrm{Spec}\,\mathrm{Symm}(\Omega_{X/A})$$

the tangent scheme of X. Also, we set

$$\mathcal{O}^{\infty}_{pq}(X) = \lim_{\rightarrow} \mathcal{O}^{r}_{pq}(X)\,.$$

Let $\pi : T(X) \to X$ be the canonical projection. Using ideas in [9], we construct a natural compatible sequence of A-derivations

$$\Theta : \mathcal{O}^{r}_{pq}(X) \to \mathcal{O}^{r}_{pq}(T(X)) \tag{44}$$

inducing an A-derivation

$$\Theta : \mathcal{O}^{\infty}_{pq}(X) \to \mathcal{O}^{\infty}_{pq}(T(X)), \tag{45}$$

respecting the filtration by orders. We call Θf the $\{\delta_p,\delta_q\}$-*tangent map* or *Fréchet derivative* of f. (In the ordinary case treated in [9] Θ was denoted by T.) The construction is local and natural, so it provides a global concept. Thus, we may assume that X is affine.

For any ring S, we denote by $S[\epsilon]$ (where $\epsilon^2=0$) the ring $S\oplus\epsilon S$ of *dual numbers* over S. Note that any prolongation sequence $S^*=\{S^r\}$ can be uniquely extended to a prolongation sequence $S^*[\epsilon]=\{S^r[\epsilon]\}$ where $\delta_p\epsilon=\epsilon$, and so $\phi_p(\epsilon)=p\epsilon$, and $\delta_q\epsilon=0$, respectively. In particular, we have a $\{\delta_p,\delta_q\}$-prolongation sequence $\mathcal{O}^{*}_{pq}(T(X))[\epsilon]=\{\mathcal{O}^{r}_{pq}(T(X))[\epsilon]\}$.

On the other hand, we have a natural inclusion $\mathcal{O}(X)\subset\mathcal{O}(T(X))$, and a natural derivation

$$d : \mathcal{O}(X) \to \mathcal{O}(T(X)) = \mathrm{Symm}(\Omega_{\mathcal{O}(X)/A}) \tag{46}$$

induced by the universal Kähler derivation $d:\mathcal{O}(X)\to\Omega_{\mathcal{O}(X)/A}$. Hence, we have an A-algebra map

$$\begin{array}{ccc} \mathcal{O}(X) & \to & \mathcal{O}(T(X))[\epsilon] \\ f & \mapsto & f+\epsilon\cdot df \end{array}.$$

By the universality property of $\mathcal{O}^{*}_{pq}(X)$, there are naturally induced ring homomorphisms

$$\mathcal{O}^{r}_{pq}(X) \to \mathcal{O}^{r}_{pq}(T(X))[\epsilon] \tag{47}$$

whose composition with the first projection

$$\begin{array}{ccc} \mathcal{O}^{r}_{pq}(T(X))[\epsilon] & \stackrel{pr_1}{\to} & \mathcal{O}^{r}_{pq}(T(X)) \\ a+\epsilon b & \mapsto & a \end{array}$$

is the identity. Composing the morphism (47) with the second projection

$$\begin{array}{ccc} \mathcal{O}^{r}_{pq}(T(X))[\epsilon] & \stackrel{pr_2}{\to} & \mathcal{O}^{r}_{pq}(T(X)) \\ a+\epsilon b & \mapsto & b \end{array},$$

we get A-derivations as in (44), which agree with each other as r varies, hence induce an A-derivation as in (45). Note that Θ restricted to $\mathcal{O}(X)$ equals d. Also the map $f \mapsto f + \epsilon\Theta f$ in (47) commutes with ϕ_p and δ_q (by universality). In particular we get that

$$\begin{array}{rcl} \Theta \circ \phi_p & = & p \cdot \phi_p \circ \Theta\,, \\ \Theta \circ \delta_q & = & \delta_q \circ \Theta\,. \end{array} \tag{48}$$

Clearly we have the following:

Proposition 5.1. *The mapping Θ is the unique A-derivation*

$$\mathcal{O}^{\infty}_{pq}(X) \to \mathcal{O}^{\infty}_{pq}(T(X))$$

extending d in (46) *and satisfying the commutation relations in* (48)*.*

Corollary 5.2. *For any A-derivation $\partial : \mathcal{O}_X \to \mathcal{O}_X$, there exists a unique derivation*

$$\partial_\infty : \mathcal{O}^{\infty}_{pq}(X) \to \mathcal{O}^{\infty}_{pq}(X) \tag{49}$$

extending ∂ and satisfying the commutation relations

$$\begin{array}{rcl} \partial_\infty \circ \phi_p & = & p \cdot \phi_p \circ \partial_\infty\,, \\ \partial_\infty \circ \delta_q & = & \delta_q \circ \partial_\infty\,. \end{array} \tag{50}$$

The first of the two conditions above says that ∂_∞ is a δ_p-derivation. The derivation ∂_∞ extends the derivation ∂_* on $\mathcal{O}^{\infty}_{p}(X)$ in [9], Definition 3.40.

Proof. The uniqueness is clear. For the proof of existence, we may assume that X is affine. By the universality property of the Kähler differentials, ∂ induces an $\mathcal{O}(X)$-module map

$$\langle \partial, \ \rangle : \Omega_{\mathcal{O}(X)/A} \to \mathcal{O}(X)$$

such that $\langle \partial, df\rangle = \partial f$ for all $f \in \mathcal{O}(X)$. By the universality property of the symmetric algebra, we get an induced $\mathcal{O}(X)$-algebra map

$$\mathcal{O}(T(X)) \to \mathcal{O}(X)\,.$$

Composing this map with the inclusion $\mathcal{O}(X) \subset \mathcal{O}^{\infty}_{pq}(X)$, we get a homomorphism

$$\mathcal{O}(T(X)) \to \mathcal{O}^{\infty}_{pq}(X)\,.$$

By the universality property of $\{\delta_p, \delta_q\}$-jet spaces, we get a $\{\delta_p, \delta_q\}$-ring homomorphism

$$\langle \partial, \ \rangle_\infty : \mathcal{O}^{\infty}_{pq}(T(X)) \to \mathcal{O}^{\infty}_{pq}(X)\,,$$

which is an $\mathcal{O}^{\infty}_{pq}(X)$- algebra homomorphism. Composing the latter with the Fréchet derivative

$$\Theta : \mathcal{O}^{\infty}_{pq}(X) \to \mathcal{O}^{\infty}_{pq}(T(X))\,,$$

we get an A-derivation ∂_∞ as in (49),

$$\partial_\infty f = \langle \partial, \Theta f\rangle_\infty\,, \quad f \in \mathcal{O}^{\infty}_{pq}(X)\,,$$

which clearly satisfies all the required properties. □

Definition 5.3. The derivation ∂_∞ in Corollary 5.2 is the *prolongation* of ∂. An *infinitesimal* ∞*-symmetry* of an element $f \in \mathcal{O}^{\infty}_{pq}(X)$ is an A-derivation $\partial : \mathcal{O}_X \to \mathcal{O}_X$ such that $\partial_\infty f = 0$.

The concept introduced above is an analogue of that of an infinitesimal symmetry in differential geometry [24]. Note that the concept of infinitesimal ∞-symmetry

does not reduce in the ordinary arithmetic case to the concept in [9], Definition 3.48. Again, the difference lies in the powers of p occurring in these two definitions.

Next we examine the image of the Fréchet derivative. For any affine X, let us define

$$\mathcal{O}^r_{pq}(T(X))^+ := \sum_{i+j=0}^{r} \sum_{f\in\mathcal{O}(X)} \mathcal{O}^r_{pq}(X)\cdot \phi_p^i\delta_q^j(df) \subset \mathcal{O}^r(T(X))\,. \tag{51}$$

Note that if $\omega_1,\dots,\omega_d$ is a basis of $\Omega_{X/A}$ (for instance, if $T_1,\dots,T_d \in \mathcal{O}(X)$ is a system of étale coordinates and $\omega_i = dT_i$), then $\mathcal{O}^r(T(X))^+$ is a free $\mathcal{O}^r_{pq}(X)$-module with basis $\{\phi_p^i\delta_q^j\omega_m \mid_{1\le m\le d, 0\le i+j\le r}\}$.

Proposition 5.4. *The image of the map* $\Theta : \mathcal{O}^r_{pq}(X) \to \mathcal{O}^r_{pq}(T(X))$ *is contained in* $\mathcal{O}^r_{pq}(T(X))^+$.

Proof. Since Θ is an A-derivation and $\mathcal{O}^r_{pq}(X)$ is topologically generated by A and elements of the form $\delta_p^i\delta_q^j f$ with $i+j\le r$, $f\in\mathcal{O}(X)$, it suffices to check that for any such i,j,f, we have that $\Theta(\delta_p^i\delta_q^j f)\in\mathcal{O}^r_{pq}(T(X))^+$.

But

$$\begin{array}{rcl}\Theta(\delta_p^i\delta_q^j f) & = & pr_2(\delta_p^i\delta_q^j(f+\epsilon df)) \\ & = & pr_2(\delta_p^i(\delta_q^j f + \epsilon\delta_q^j(df)))\,.\end{array}$$

By [9], Lemma 3.34, the latter has the form

$$\sum_{l=0}^{i} \Lambda_{il}(\delta_q^j f, \delta_p\delta_q^j f,\dots,\delta_p^{i-1}\delta_q^j f)\phi_p^l\delta_q^j(df)$$

where Λ_{il} are polynomials with A-coefficients, and we are done. □

Remark 5.5. The Fréchet derivative is functorial with respect to pull back in the following sense. Let $f:\hat{Y}\to\hat{X}$ be a morphism of p-adic formal schemes between the p-adic completions of two schemes, X and Y, of finite type over A. Cf. Remark 3.9. Then we have a natural commutative diagram

$$\begin{array}{ccc} \mathcal{O}^r_{pq}(X) & \stackrel{f^*}{\to} & \mathcal{O}^r_{pq}(Y) \\ \Theta\downarrow & & \downarrow\Theta \\ \mathcal{O}^r_{pq}(T(X)) & \stackrel{f^*}{\to} & \mathcal{O}^r_{pq}(T(Y)) \end{array}\,. \tag{52}$$

5.2. **Fréchet symbol.** We assume in what follows that X has relative dimension 1 over A. Let $f\in\mathcal{O}^r_{pq}(X)$, and let ω be a basis of $\Omega_{X/A}$. Then we can write

$$\Theta f = \theta_{f,\omega}(\phi_p,\delta_q)\omega = \sum a_{ij}\phi_p^i\delta_q^j\omega\,, \tag{53}$$

where $\theta_{f,\omega}$ is a polynomial

$$\theta_{f,\omega} = \theta_{f,\omega}(\xi_p,\xi_q) = \sum a_{ij}\xi_p^i\xi_q^j \in \mathcal{O}^r_{pq}(X)[\xi_p,\xi_q]\,.$$

If, in addition, we have given an A-point $P\in X(A)$, by the universality property of $\{\delta_p,\delta_q\}$-jet spaces, we obtain a naturally induced lift $P^r\in J^r_{pq}(X)(A)$ of P. For any $g\in\mathcal{O}^r_{pq}(X)=\mathcal{O}(J^r_{pq}(X))$, we denote by $g(P)\in A$ the image of g under the "evaluation" homomorphism $\mathcal{O}^r_{pq}(X)\to A$ induced by P^r. Then we may consider the polynomial

$$\theta_{f,\omega,P}(\xi_p,\xi_q) = \sum a_{ij}(P)\xi_p^i\xi_q^j \in A[\xi_p,\xi_q]\,.$$

Definition 5.6. The polynomial $\theta_{f,\omega}$ is the *Fréchet symbol* of f with respect to ω. The polynomial $\theta_{f,\omega,P}$ is the *Fréchet symbol* of f at P with respect to ω.

The polynomials $\theta_{f,\omega}$ and $\theta_{f,\omega,P}$ have a certain covariance property with respect to ω, which we explain next.

Definition 5.7. Let B be any $\{\delta_p,\delta_q\}$-ring in which p is a non-zero divisor. We define the (right) action of the group $B^\times$ on the ring $B[\xi_p,\xi_q]$ as follows. If $b\in B^\times$ and $\theta\in B[\xi_p,\xi_q]$, then $\theta\cdot b\in B[\xi_p,\xi_q]$ is the unique polynomial with the property that for any $\{\delta_p,\delta_q\}$-ring extension C of B, and any $x\in C$, we have

$$((\theta\cdot b)(\phi_p,\delta_q))(x)=\theta(\phi_p,\delta_q)\cdot(bx)\,.$$

Now taking B to be either $\mathcal{O}^\infty_{pq}(X)$ or A, it is easy to see that

$$\begin{array}{rcl}\theta_{f,g\omega} & = & \theta_{f,\omega}\cdot g^{-1}\\ \theta_{f,g\omega,P} & = & \theta_{f,\omega,P}\cdot (g(P))^{-1}\end{array}\,. \tag{54}$$

In particular, $\theta_{f,\omega,P}$ only depends on the image $\omega(P)$ of ω in the *cotangent space of X at P*, $\Omega_{X/A}\otimes_{\mathcal{O}(X),P}A$, where A here is viewed as an $\mathcal{O}(X)$-module via the evaluation map $P:\mathcal{O}(X)\to A$.

Remark 5.8. The proof of Corollary 5.2 and (53) show that if ω is a basis of $\Omega_{X/A}$ and $\partial:\mathcal{O}_X\to\mathcal{O}_X$ is an A-derivation, then for any $f\in\mathcal{O}^\infty_{pq}(X)$ we have that

$$\partial_\infty f=\langle\partial,\theta_{f,\omega}(\phi_p,\delta_q)\omega\rangle_\infty=\theta_{f,\omega}(\phi_p,\delta_q)(\langle\partial,\omega\rangle), \tag{55}$$

where $\theta_{f,\omega}$ is the Fréchet symbol of f with respect to ω.

Our next proposition relates Fréchet symbols of $\{\delta_p,\delta_q\}$-characters to the Picard-Fuchs symbol; cf. Definition 4.4. This will be useful later, as in the applications to come, the Picard-Fuchs symbols will be easy to compute.

We let G be either $\mathbb{G}_a$, or $\mathbb{G}_m$, or an elliptic curve over A.

Proposition 5.9. *Assume that ω is an A-basis for the invariant 1-forms on G, and let ψ be a $\{\delta_p,\delta_q\}$-character of G. Let $\theta=\theta_{\psi,\omega}\in\mathcal{O}^\infty_{pq}[\xi_p,\xi_q]$ be the Fréchet symbol of ψ with respect to ω, and let $\sigma\in A[\xi_p,\xi_q]$ be the Picard-Fuchs symbol of ψ with respect to an étale coordinate T at the origin 0 such that $\omega(0)=dT$. Then*

$$\sigma^{(p)}=p\theta\,.$$

Remark 5.10.

1) The relation $\sigma^{(p)}=p\theta$ reads $\sigma(p\xi_p,\xi_q)=p\theta(\xi_p,\xi_q)$.
2) The expression above implies that, in particular, $\theta\in A[\xi_p,\xi_q]$, and $\Theta\psi=\frac{1}{p}\sigma(p\phi_p,\delta_q)\omega$.
3) If $\mathrm{Lie}(G)$ denotes the A-module of invariant A-derivations of $\mathcal{O}_G$, and if $\partial\in\mathrm{Lie}(G)$, by (55) we then have that

$$\partial_\infty\psi=\frac{1}{p}\sigma(p\phi_p,\delta_q)(\langle\partial,\omega\rangle)\in A\,. \tag{56}$$

In particular, ∂ is an infinitesimal ∞-symmetry of ψ if, and only if, the element $u:=\langle\partial,\omega\rangle\in A=\mathbb{G}_a(A)$ is a solution to the $\{\delta_p,\delta_q\}$-character of $\mathbb{G}_a=\mathrm{Spec}\,A[y]$ defined by $\frac{1}{p}\sigma(p\phi_p,\delta_q)y\in A\{y\}$.

Proof of Proposition 5.9. Consider the diagram in (52) with $X = G$, $Y = \mathbb{G}_a$, and $f = e(pT)$. By the definition of the Picard-Fuchs symbol, if $\sigma = \sum a_{ij}\xi_p^i\xi_q^j$ then

$$e(pT)^*\psi = \sum a_{ij}\phi_p^i\delta_q^j T\,.$$

Applying the Fréchet derivative to this identity, and using the commutativity of the diagram (52), we get that

$$e(pT)^*\Theta\psi = \Theta(e(pT)^*\psi) = \Theta(\sum a_{ij}\phi_p^i\delta_q^j T) = \sum a_{ij}p^i\phi_p^i\delta_q^j(dT)\,. \tag{57}$$

On the other hand, we may write $\theta_{\psi,\omega} = \sum b_{ij}\xi_p^i\xi_q^j$, with $b_{ij} \in \mathcal{O}^r_{pq}(G)$. By the definition of the Fréchet symbol, we have $\Theta\psi = \sum b_{ij}\phi_p^i\delta_q^j\omega$, and since $\omega(0) = dT$, we have that $\omega = dl(T)$, where $l(T)$ is the compositional inverse of $e(T)$. Therefore, $e(pT)^*\omega = d(l(e(pT))) = pdT$. We obtain that

$$e(pT)^*\Theta\psi = e(pT)^*\left(\sum b_{ij}\phi_p^i\delta_q^j\omega\right) = p\sum b_{ij}\phi_p^i\delta_q^j(dT)\,. \tag{58}$$

The identities (57) and (58) imply that $\left(\sum a_{ij}p^i\phi_p^i\delta_q^j\right)(dT) = p\left(\sum b_{ij}\phi_p^i\delta_q^j\right)(dT)$, and this finishes the proof. $\square$

5.3. **Euler-Lagrange equations.** The Fréchet symbol can be also used to introduce an Euler-Lagrange formalism in our setting. In order to explain this, we begin by making the following definition.

Definition 5.11. Let B be any $\{\delta_p, \delta_q\}$-ring in which p is a non-zero divisor. For any $r \geq 1$, we denote by $B[\xi_p, \xi_q]_r$ the submodule of the polynomial ring $B[\xi_p, \xi_q]$ consisting of all polynomials of degree $\leq r$. The *adjunction map*

$$Ad^r : B[\xi_p, \xi_q]_r \to B[\xi_p, \xi_q][1/p]$$

is defined by

$$Ad^r(\sum_{ij} b_{ij}\xi_p^i\xi_q^j) := \sum_{ij}(-1)^j p^{-ij}\xi_p^{r-i}\xi_q^j \cdot b_{ij}\,. \tag{59}$$

This map induces an *adjunction map*

$$ad^r : B[\xi_p, \xi_q]_r \to B[1/p]$$

by

$$ad^r(Q) := (Ad^r(Q)(\phi_p, \delta_q))(1)\,,$$

which is explicitly given by

$$ad^r(\sum_{ij} b_{ij}\xi_p^i\xi_q^j) := \sum_{ij}(-1)^j p^{-ij}\phi_p^{r-i}\delta_q^j b_{ij}\,. \tag{60}$$

For any $Q \in B[\xi_p, \xi_q]_r$, we have

$$ad^{r+1}(Q) = (ad^r(Q))^\phi\,.$$

This adjunction map is a hybrid between the "familiar" adjunction map for usual linear differential operators with respect to δ_q (as encountered after the usual integration by parts argument in the calculus of variations), and the adjunction map for δ_p-characters (as defined in [9]). The definition of the adjunction map above might seem ad hoc, and somewhat complicated, but it is justified by the following covariance property.

Lemma 5.12. *For any $Q \in B[\xi_p, \xi_q]_r$ and $b \in B$, we have that*

$$ad^r(Q \cdot b) = b^{\phi^r} \cdot ad^r(Q)\,.$$

Proof. This follows from a direct computation. □

The mapping ad^r is related (but does not coincide) with the mapping ad_r defined in [9], p. 92. Indeed, the powers of p in the definitions of ad^r and ad_r are different.

Returning to our geometric setting, let X/A be a smooth scheme of relative dimension 1, let $f \in \mathcal{O}^r_{pq}(X)$, and let $\partial : \mathcal{O}_X \to \mathcal{O}_X$ be an A-derivation. We define an element $\epsilon^r_{f,\partial} \in \mathcal{O}^\infty_{pq}(X)[1/p]$ as follows. Let us assume first that X is affine and $\Omega_{X/A}$ is free with basis ω. We let B in the discussion above be the $\{\delta_p, \delta_q\}$-ring $\mathcal{O}^\infty_{pq}(X)$. We consider the Fréchet symbol $\theta_{f,\omega} \in \mathcal{O}^\infty_{pq}(X)[\xi_p, \xi_q]_r$, and its image under ad^r, $ad^r(\theta_{f,\omega}) \in \mathcal{O}^\infty_{pq}(X)[1/p]$. Then we set

$$\epsilon^r_{f,\partial} := \langle \partial, \omega\rangle^{\phi^r} \cdot ad^r(\theta_{f,\omega}) \in \mathcal{O}^\infty_{pq}(X)[1/p]\,. \tag{61}$$

By Lemma 5.12 and (54), $\epsilon^r_{f,\partial}$ does not depend on the choice of ω. Therefore, this definition globalizes to one in the case when X is not necessarily affine and $\Omega_{X/A}$ is not necessarily free.

Definition 5.13. We say that the element $\epsilon^r_{f,\partial}$ is the *Euler-Lagrange equation* attached to the *Lagrangian* f and the *vector field* ∂.

Definition 5.14. An *energy function* of order r on G is a $\{\delta_p, \delta_q\}$-morphism $H : G \to \mathbb{A}^1$ that can be written as

$$H = \sum_{ij} h_{ij}\psi_i\psi_j\,, \tag{62}$$

where $h_{ij} \in A$ and ψ_i are $\{\delta_p, \delta_q\}$-characters of G of order r.

Proposition 5.15. *If H is an energy function of order r on G and $\partial : \mathcal{O}_G \to \mathcal{O}_G$ is an A-derivation that constitutes a basis of* $\mathrm{Lie}(G)$*, then the Euler-Lagrange equation $\epsilon^r_{H,\partial}$ is a K-multiple of a $\{\delta_p, \delta_q\}$-character.*

Proof. Let us assume that H is an in (62), and let ω be an A-basis of the invariant 1-forms on G. Let

$$\theta_{\psi_i,\omega} = \sum_{mn} a_{imn}\xi_p^m \xi_q^n\,. \tag{63}$$

Then

$$\begin{array}{rcl} \Theta H & = & \sum_{ij} h_{ij}(\psi_i \Theta\psi_j + \psi_j\Theta\psi_i) \\ & = & \sum_{mnij}[h_{ij}(a_{jmn}\psi_i + a_{imn}\psi_j)]\phi_p^m\delta_q^n\omega\,. \end{array}$$

Hence

$$\epsilon^r_{H,\partial} = \langle\partial,\omega\rangle^{\phi^r} \cdot \sum_{mnij} (-1)^n p^{-mn}\phi_p^{r-m}\delta_q^n[h_{ij}(a_{jmn}\psi_i + a_{imn}\psi_j)]\,, \tag{64}$$

and we are done. □

Definition 5.16. A *boundary element* in $\mathcal{O}^\infty_{pq}(X) \otimes K$ is an element of the form $\delta_q a + \phi_p b - b$, for some $a, b \in \mathcal{O}^\infty_{pq}(X) \otimes K$.

The following can be interpreted as an analogue of Noether's Theorem in mechanics [24]. It is a hybrid between the usual Noether Theorem and the "arithmetic Noether Theorem" in [9]. We state it in the affine case.

Proposition 5.17. *Let X be an affine smooth scheme over A of relative dimension 1, and let $\partial : \mathcal{O}_X \to \mathcal{O}_X$ be an A-derivation that is a basis for the $\mathcal{O}(X)$-module of all A-derivations. Then, for any $f \in \mathcal{O}^r_{pq}(X)$ we that $\epsilon^r_{f,\partial} - \partial_\infty f \in \mathcal{O}^\infty_{pq}(X) \otimes K$ is a boundary element. In particular, if ∂ is an infinitesimal ∞-symmetry of f, then $\epsilon^r_{f,\partial}$ is a boundary element.*

Proof. Let ω be a basis of $\Omega_{X/A}$, and let $\theta_{f,\omega} = \sum b_{ij}\xi_p^i\xi_q^j$ for $b_{ij} \in \mathcal{O}^\infty_{pq}(X)$. We set $v := \langle \partial, \omega\rangle \in \mathcal{O}(X)^\times$ (in what follows we might assume that $v = 1$, but this would not simplify the computation), and denote by $\equiv$ the congruence relation in $\mathcal{O}^\infty_{pq}(X) \otimes K$ modulo the group of boundary elements. Then we have

$$\begin{array}{rcl} \partial_\infty f & = & \sum b_{ij}\phi_p^i\delta_q^j v = \sum b_{ij}p^{-ij}\delta_q^j\phi_p^i v \\ & \equiv & \sum(-1)^j p^{-ij}(\delta_q^j b_{ij})(\phi_p^i v)\,. \end{array}$$

We also we have that

$$\begin{array}{rcl} \epsilon^r_{f,\partial} & = & \sum(-1)^j p^{-ij}(\phi_p^r v)(\phi_p^{r-i}\delta_q^j b_{ij}) \\ & = & \sum(-1)^j p^{-ij}\phi_p^{r-i}[(\delta_q^j b_{ij})(\phi_p^i v)]\,. \end{array}$$

Consequently,

$$\epsilon^r_{f,\partial} - \partial_\infty f \equiv \sum(-1)^j p^{-ij}(\phi_p^{r-i} - 1)[(\delta_q^j b_{ij})(\phi_p^i v)] \equiv 0$$

because $\phi_p^k - 1$ is divisible by $\phi_p - 1$ in $A[\phi_p]$. $\square$

6. Additive group

In this section we prove our main results about $\{\delta_p, \delta_q\}$-characters and their space of solutions in the case where G is the additive group.

Let $\mathbb{G}_a = \operatorname{Spec} A[y]$ be the additive group over our fixed $\{\delta_p, \delta_q\}$-ring A. We equip $\mathbb{G}_a$ with the invariant 1-form

$$\omega := dy\,.$$

Proposition 6.1. *The A-module $\mathbf{X}^r_{pq}(\mathbb{G}_a)$ of $\{\delta_p, \delta_q\}$-characters of order r on $\mathbb{G}_a$ is free with basis*

$$\{\phi_p^i\delta_q^j y\ |_{0\le i+j\le r}\}\,.$$

Hence the $A[\phi_p, \delta_q]$-module $\mathbf{X}^\infty_{pq}(\mathbb{G}_a)$ of $\{\delta_p, \delta_q\}$-characters of $\mathbb{G}_a$ is free of rank one with basis y.

Proof. Same argument as in the proof of Lemma 4.2. $\square$

Throughout the rest of this section, we let $A = R$. In particular, the notation and discussion in Example 2.16 applies to $\mathbb{G}_a$ over R.

By Proposition 6.1, any $\{\delta_p, \delta_q\}$-character of $\mathbb{G}_a$ can be written uniquely as

$$\psi_a := \psi_a^\mu := \mu(\phi_p, \delta_q)y \in R[y, Dy, \ldots, D^n y]\widehat{\ }, \tag{65}$$

where

$$\mu(\xi_p, \xi_q) = \sum \mu_{ij}\xi_p^i\xi_q^j \in R[\xi_p, \xi_q]$$

is a polynomial. Note that the Picard-Fuchs symbol $\sigma(\xi_p, \xi_q)$ of ψ_a with respect to the étale coordinate $T = y$ is given by

$$\sigma(\xi_p, \xi_q) = p\mu(\xi_p, \xi_q).$$

The Fréchet symbol of ψ_a with respect to $\omega = dy$ is

$$\theta(\xi_p, \xi_q) = \mu(p\xi_p, \xi_q).$$

Definition 6.2. We say that $\mu(\xi_p,\xi_q)$ is the *characteristic polynomial* of the character ψ_a. We say that the $\{\delta_p,\delta_q\}$-character ψ_a is *non-degenerate* if $\mu(0,0)\in R^\times$. Given a non-degenerate character ψ_a, we say that $\kappa\in\mathbb{Z}$ is a *characteristic integer* of ψ_a if $\mu(0,\kappa)=0$. (Note that any characteristic integer of a non-degenerate $\{\delta_p,\delta_q\}-$ character ψ_a must be coprime to p.) We say that $\kappa\in\mathbb{Z}$ is a *totally non-characteristic* integer if $\kappa\not\equiv 0 \bmod p$ and $\mu(0,\kappa)\not\equiv 0 \bmod p$. We denote by $\mathcal{K}$ the set of all characteristic integers of ψ_a, and set $\mathcal{K}_\pm := \mathcal{K}\cap\mathbb{Z}_\pm$. We denote by $\mathcal{K}'$ the set of all totally non-characteristic integers. For all $0\neq\kappa\in\mathbb{Z}$ and $\alpha\in R$, we define the *basic series* of ψ_a by

$$
\begin{array}{rcl}
u_{a,\kappa,\alpha} := u^\mu_{a,\kappa,\alpha} & := & \sum_{n\geq 0} b_{n,\kappa}\phi_p^n(\alpha q^\kappa) \\
 & = & \sum_{n\geq 0} b_{n,\kappa}\alpha^{\phi^n} q^{\kappa p^n} \\
 & = & \alpha q^\kappa + \cdots \in q^{\pm 1}R[[q^{\pm 1}]],
\end{array}
\tag{66}
$$

where $\{b_{n,\kappa}\}_{n\geq 0}$ is the sequence of elements in R defined inductively by $b_{0,\kappa}=1$,

$$
b_{n,\kappa} := -\frac{\sum_{s=1}^n \left(\sum_{j\geq 0}\mu_{sj}\kappa^j p^{j(n-s)}\right) b^{\phi^s}_{n-s,\kappa}}{\sum_{j\geq 0}\mu_{0j}\kappa^j p^{jn}}\,,\ n\geq 1\,.
\tag{67}
$$

In this last expression, the denominator is congruent to $\mu(0,0)$ mod p (and is therefore an element of $R^\times$).

The next Lemma intuitively says that the two collections of series $\{u_{a,\kappa,1}\mid \kappa\neq 0\}$ and $\{q^\kappa\mid\kappa\neq 0\}$ "diagonalize" ψ_a.

Lemma 6.3. *For all $0\neq\kappa\in\mathbb{Z}$ and $\alpha\in R$ we have $\psi_a u_{a,\kappa,\alpha}=\mu(0,\kappa)\cdot\alpha q^\kappa$.*

Proof. The desired result follows from the following computation:

$$
\begin{array}{rcl}
\psi_a u_{a,\kappa,\alpha} & = & \sum_{i,j,m\geq 0}\mu_{ij}\phi_p^i\delta_q^j(b_{m,\kappa}\phi_p^m(\alpha q^\kappa)) \\
 & = & \sum_{i,j,m\geq 0}\mu_{ij} b^{\phi^i}_{m,\kappa}\kappa^j p^{jm}\phi_p^{i+m}(\alpha q^\kappa) \\
 & = & \sum_{n\geq 0}\left(\sum_{s\geq 0}\left(\sum_{j\geq 0}\mu_{sj}\kappa^j p^{j(n-s)}\right) b^{\phi^s}_{n-s,\kappa}\right)\phi_p^n(\alpha q^\kappa) \\
 & = & \left(\sum_{j\geq 0}\mu_{0j}\kappa^j\right)\alpha q^\kappa \\
 & = & \mu(0,\kappa)\alpha q^\kappa\,.
\end{array}
\tag{68}
$$

□

For any series $u\in R[[q^{\pm 1}]]$ let $\bar{u}\in k[[q^{\pm 1}]]$ denote the reduction of u mod p; we recall that $k=R/pR$. Also we denote by

$$
\bar{\mu}(\xi_p,\xi_q)=\sum\bar{\mu}_{ij}\xi_p^i\xi_q^j\in k[\xi_p,\xi_q]
$$

the reduction mod p of the characteristic polynomial $\mu(\xi_p,\xi_q)$. It is convenient to introduce the following terminology.

Definition 6.4. We say that a polynomial $\mu\in R[\xi_p,\xi_q]$ is *unmixed* if $\bar{\mu}_{ij}=0$ for $ij\neq 0$ and there exists $i\neq 0$ such that $\bar{\mu}_{i0}\neq 0$. Equivalently, μ is unmixed if

$$
\bar{\mu}(\xi_p,\xi_q)=\bar{\mu}(\xi_p,0)+\bar{\mu}(0,\xi_q)-\bar{\mu}(0,0)\,,
$$

and

$$
\bar{\mu}(\xi_p,0)\neq\bar{\mu}(0,0)\,.
$$

Definition 6.5. We say that $S \subset \mathbb{Z}_+\backslash\{0\}$ is *short* if

$$\frac{\max S}{\min S} < \frac{p}{2}\,.$$

We say that $S \subset \mathbb{Z}_-\backslash\{0\}$ is *short* if the set $-S$ is short. In particular, a set $S \subset \mathbb{Z}\backslash\{0\}$ consisting of a single element is short.

Lemma 6.6. *Let $S \subset \mathbb{Z}_\pm\backslash p\mathbb{Z}_\pm$ be a non-empty finite set of either positive or negative integers, and let*

$$u := \sum_{\kappa\in S} u_{a,\kappa,\alpha_\kappa}\,,$$

where $\alpha_\kappa \in R$, not all of them in pR. Then the following hold:

1. *$\bar{u}$ is integral over $k[q^{\pm 1}]$, and the field extension $k(q) \subset k(q,\bar{u})$ is Abelian with Galois group killed by p.*
2. *If μ is unmixed, then $\bar{u} \notin k(q)$.*
3. *If μ is unmixed and S is short, then u is transcendental over $K(q)$.*

Proof. Let us assume that $S \subset \mathbb{Z}_+$. The case $S \subset \mathbb{Z}_-$ is treated in a similar manner.

We prove assertion 1. By (67), for $n \geq 1$, we have that

$$\begin{array}{rcl} \bar{b}_{n,\kappa} & = & -(\bar{\mu}_{00})^{-1}\left(\sum_{s=1}^{n-1}\bar{\mu}_{s0}\bar{b}_{n-s,\kappa}^{p^s} + \sum_{j\geq 0}\bar{\mu}_{nj}\bar{\kappa}^j\right) \\ & = & -(\bar{\mu}_{00})^{-1}\left(\sum_{s\geq 1}\bar{\mu}_{s0}\bar{b}_{n-s,\kappa}^{p^s} + \sum_{j\geq 1}\bar{\mu}_{nj}\bar{\kappa}^j\right)\,. \end{array}$$

Also

$$\bar{u} = \sum_{\kappa\in S}\sum_{n\geq 0}\bar{b}_{n,\kappa}\bar{\alpha}_\kappa^{p^n}q^{\kappa p^n}\,.$$

Let us denote by $F_p : k \to k$ the p-th power Frobenius map. Consider the polynomial $g(q) \in k[q]$ given by

$$\begin{array}{rcl} g(q) & = & \sum_{\kappa\in S}[\bar{\mu}(F_p,\kappa) - \bar{\mu}(F_p,0) - \bar{\mu}(0,\kappa)](\alpha q^\kappa) \\ & = & \sum_{\kappa\in S}\sum_{n\geq 0}\sum_{j\geq 1}\bar{\mu}_{nj}\bar{\kappa}^j\bar{\alpha}_\kappa^{p^n}q^{\kappa p^n} - \sum_{\kappa\in S}\sum_{j\geq 0}\bar{\mu}_{0j}\bar{\kappa}^j\bar{\alpha}_\kappa q^\kappa\,, \end{array}$$

and define the polynomial $G(t) \in k(q)[t]$ by

$$G(t) := \sum_{s\geq 0}\bar{\mu}_{s0}t^{p^s} + g(q)\,.$$

We have that $G(t)$ has an invertible leading coefficient, is separable ($dG/dt = \bar{\mu}_{00}$), and has $\bar{u}$ as a root:

$$\begin{array}{rcl} G(\bar{u}) & = & \sum_{\kappa\in S}\sum_{s\geq 0}\sum_{n\geq 0}\bar{\mu}_{s0}\bar{b}_{n,\kappa}^{p^s}\bar{\alpha}_\kappa^{p^{n+s}}q^{\kappa p^{n+s}} \\ & & +\sum_{\kappa\in S}\sum_{n\geq 0}\sum_{j\geq 1}\bar{\mu}_{nj}\bar{\kappa}^j\bar{\alpha}_\kappa^{p^n}q^{\kappa p^n} - \sum_{\kappa\in S}\sum_{j\geq 0}\bar{\mu}_{0j}\bar{\kappa}^j\bar{\alpha}_\kappa q^\kappa \\ & = & \sum_{\kappa\in S}\sum_{n\geq 0}\left(\sum_{s\geq 0}\bar{\mu}_{s0}\bar{b}_{n-s,\kappa}^{p^s} + \sum_{j\geq 1}\bar{\mu}_{nj}\bar{\kappa}^j\right)\bar{\alpha}_\kappa^{p^n}q^{\kappa p^n} \\ & & -\sum_{\kappa\in S}\sum_{j\geq 0}\bar{\mu}_{0j}\bar{\kappa}^j\bar{\alpha}_\kappa q^\kappa \\ & = & 0\,. \end{array}$$

The difference of any two roots of $G(t)$ is in k. Hence, $k(q,\bar{u})$ is Galois over $k(q)$ and its Galois group Σ embeds into k via the map

$$\begin{array}{rcl} \Sigma & \to & k \\ \sigma & \mapsto & \sigma\bar{u} - \bar{u}\,. \end{array}$$

We prove assertion 2. In this case we have

$$g(q) = -\bar{\mu}(0,0) \cdot \sum_{\kappa \in S} \bar{\alpha}_\kappa q^\kappa \, ,$$

and $G(t)$ has degree p^e in t for some $e \geq 1$. Assume that $\bar{u} \in k(q)$. Since $\bar{u}$ is integral over $k[q]$ it follows that $\bar{u} \in k[q]$. Let d be the degree of $\bar{u}$. Since $G(\bar{u}) = 0$, we see that $d \geq 1$. Since the integers in S are not divisible by p, the coefficient of q^{dp^e} in the polynomial $G(\bar{u}) \in k[q]$ is non-zero, a contradiction.

We prove assertion 3. Let $\kappa_1 < \kappa_2 < \cdots < \kappa_s$ be the integers in S. By our assumption, there is a real $\epsilon > 0$ such that $(2+\epsilon)\kappa_s < \kappa_1 p$. We will show that

$$u(q^{-1}) = \sum_{j=1}^{s} \sum_{n \geq 0} b_{n,\kappa_j} \alpha_{\kappa_j}^{\phi^n} q^{-\kappa_j p^n} \in K((q^{-1}))$$

is transcendental over $K(q)$.

For any $\varphi = \sum_{n=-\infty}^{f} c_n q^n \in K((q^{-1}))$ with $c_f \geq 0$, set $|\varphi| := e^f$. Also, set $|0| = 0$. Roth's theorem for characteristic zero function fields [31] states that if φ is algebraic over $K(q)$ then, for any $\epsilon > 0$, the inequality

$$0 < |\varphi - P/Q| < |Q|^{-2-\epsilon}$$

has only finitely many solutions P/Q with $P, Q \in K[q]$. On the other hand, for any n, we have

$$\begin{array}{rcl} 0 & < & |u(q^{-1}) - \sum_{j=1}^{s} \sum_{n=0}^{N} b_{n,\kappa_j} \alpha_{\kappa_j}^{\phi^n} q^{-\kappa_j p^n}| \\ & \leq & e^{-\kappa_1 p^{n+1}} \\ & < & (e^{\kappa_s p^n})^{-2-\epsilon} \\ & = & |q^{\kappa_s p^n}|^{-2-\epsilon} \, . \end{array}$$

It follows that $u(q^{-1})$ is transcendental over $K(q)$, and this completes the proof. $\square$

Remark 6.7.

1) If $\mu_{ij} \in pR$ for $i \geq 1$, then $u_{a,\kappa,\alpha} \in R[q,q^{-1}]\hat{}$.
2) The mapping

$$\begin{array}{ccc} R & \rightarrow & R[[q^{\pm 1}]] \\ \alpha & \mapsto & u_{a,\kappa,\alpha} \end{array}$$

is an injective group homomorphism.
3) Recall that, attached to a δ_q-character ψ_q we defined in (31) operators B^0_κ and $B^0_\pm$. Let ψ_q be, in our case, the identity; hence, for $0 \neq \kappa \in \mathbb{Z}$,

$$\begin{array}{ccc} R[[q^{\pm 1}]] & \stackrel{B^0_\kappa}{\rightarrow} & R \\ B^0_\kappa \left(\sum a_n q^n\right) & = & a_\kappa , \end{array}$$

and

$$\begin{array}{ccc} R[[q^{\pm 1}]] & \stackrel{B^0_\pm}{\rightarrow} & R^{\rho_\pm} \\ B^0_\pm \left(\sum a_n q^n\right) & = & (a_\kappa)_{\kappa \in \mathcal{K}_\pm} . \end{array}$$

Note that if $\kappa_1, \kappa_2 \in \mathbb{Z} \backslash p\mathbb{Z}$, we have

$$B^0_{\kappa_1} u_{a,\kappa_2,\alpha} = \alpha \cdot \delta_{\kappa_1 \kappa_2} , \tag{69}$$

where $\delta_{\kappa_1 \kappa_2}$ is the Kronnecker delta.

4) We have the identity

$$\delta_q u^{\mu}_{a,\kappa,\alpha} = \kappa \cdot u^{\mu^{(p)}}_{a,\kappa,\alpha}\,, \tag{70}$$

where we recall that $\mu^{(p)}(\xi_p,\xi_q) := \mu(p\xi_p,\xi_q)$.

5) We have the identity

$$u_{a,\kappa,\zeta^\kappa\alpha}(q) = u_{a,\kappa,\alpha}(\zeta q) \tag{71}$$

for all $\zeta \in \mu(R)$; this holds because $\zeta^\phi = \zeta^p$. In particular, if $\alpha = \sum_{i=0}^\infty m_i\zeta_i^\kappa$, $\zeta_i \in \mu(R)$, $m_i \in \mathbb{Z}$, $v_p(m_i) \to \infty$, then

$$u_{a,\kappa,\alpha}(q) = \sum_{i=0}^\infty m_i u_{a,\kappa,1}(\zeta_i q)\,.$$

Thus, if $f \in \mathbb{Z}\mu(R)\widehat{\ }$ is such that $(f^{[\kappa]})^\sharp = \alpha \in R$, then $u_{a,\kappa,\alpha}$ can be expressed using convolution:

$$u_{a,\kappa,\alpha} = f \star u_{a,\kappa,1}\,.$$

Note that $\{u_{a,\kappa,\alpha} \mid \alpha \in R\}$ is a $\mathbb{Z}\mu(R)\widehat{\ }$-module (under convolution). If $\kappa \in \mathbb{Z}\backslash p\mathbb{Z}$, this module structure comes from an R-module structure, still denoted by $\star$, by a base change via the surjective homomorphism

$$\mathbb{Z}\mu(R)\widehat{\ } \stackrel{[\kappa]}{\to} \mathbb{Z}\mu(R)\widehat{\ } \stackrel{\sharp}{\to} R$$

(see (34)), and the R-module $\{u_{a,\kappa,\alpha} \mid \alpha \in R\}$ is free with basis $u_{a,\kappa,1}$. Hence, for $g \in \mathbb{Z}\mu(R)\widehat{\ }$, $\beta = (g^{[\kappa]})^\sharp$, we have that

$$\beta \star u_{a,\kappa,\alpha} = g \star u_{a,\kappa,\alpha}\,.$$

In particular,

$$u_{a,\kappa,\alpha} = \alpha \star u_{a,\kappa,1}\,.$$

6) We have the following "rationality" property: if $\alpha, \mu_{ij} \in \mathbb{Z}_{(p)}$, then $u_{a,\kappa,\alpha} \in \mathbb{Z}_{(p)}[[q^{\pm 1}]]$.

Definition 6.8. We say that the mapping

$$\begin{array}{ccc} R & \to & R[[q]] \\ \alpha & \mapsto & v_\alpha \end{array}$$

is a *pseudo δ_p-polynomial map* if for any integer $n \geq 0$ there exists an integer $r_n \geq 0$ and a polynomial $P_n \in R[x_0,x_1,\ldots,x_{r_n}]$ such that, for all $\alpha \in R$ we have that

$$v_\alpha = \sum P_n(\alpha,\delta_p\alpha,\ldots,\delta_p^{r_n}\alpha)q^n\,. \tag{72}$$

If X is a scheme over $R[[q]]$, then a map $R \to X(R[[q]])$ is said to be a *pseudo δ_p-polynomial map* if there exists an open subscheme $U \subset X$, and a closed embedding $U \subset \mathbb{A}^n$ such that the image of $R \to X(R[[q]])$ is contained in $U(R[[q]])$, and the maps

$$R \to U(R[[q]]) \subset \mathbb{A}^N(R[[q]]) = R[[q]]^N \stackrel{pr_i}{\to} R[[q]],$$

are pseudo δ_p-polynomial. Here pr_i are the various projections.

Similarly, a mapping

$$\begin{array}{ccc} R & \to & R[[q^{-1}]] \\ \alpha & \mapsto & v_\alpha \end{array}$$

is said to be a *pseudo δ_p-polynomial map* if for any integer $n \leq 0$ there exists an integer $r_n \geq 0$ and a polynomial $P_n \in R[x_0,x_1,\ldots,x_{r_n}]$ such that (72) holds

for all $\alpha \in R$, and given a scheme X over $R[[q^{-1}]]$, a *pseudo δ_p-polynomial* map $R \to X(R[[q^{-1}]])$ is defined as above, with the rôle of $R[[q]]$ now being played by $R[[q^{-1}]]$.

The prefix *pseudo* was included in order to suggest an analogy with "differential operators of infinite order." This is not to be confused with the pseudo-differential operators in micro-local analysis.

Example 6.9. The basic series mappings

$$\begin{array}{ccc} R & \to & R[[q^{\pm 1}]] \\ \alpha & \mapsto & u_{a,\kappa,\alpha} \end{array}$$

are pseudo δ_p-polynomial maps. In particular, when interpreted as mapping $R \to \mathbb{G}_a(R[[q^{\pm 1}]])$, they are pseudo δ_p-polynomial maps.

The following example is as elementary as they come. More interesting ones will be given later on, while studying $\mathbb{G}_m$ and elliptic curves.

Theorem 6.10. *Let ψ_a be a non-degenerate $\{\delta_p, \delta_q\}$-character of $\mathbb{G}_a$, and let $\mathcal{U}_*$ be the corresponding groups of solutions. Let $\mathcal{K}$ be the set of characteristic integers, and $u_{a,\kappa,\alpha}$ be the basic series. Then the following hold:*

1) *If $\mathcal{K} = \emptyset$, then $\mathcal{U}_{\leftarrow} = \mathcal{U}_{\rightarrow} = \mathcal{U}_0$.*
2) *We have*
$$\mathcal{U}_{\pm 1} \quad = \quad \bigoplus_{\kappa \in \mathcal{K}_{\pm}} \{u_{a,\kappa,\alpha} \mid \alpha \in R\} \, ,$$
where $\oplus$ denotes internal direct sum. In particular, $\mathcal{U}_{\pm 1}$ are free R-modules under convolution, with bases $\{u_{a,\kappa,1} \mid \kappa \in \mathcal{K}_{\pm}\}$, respectively.
3) *$\mathcal{U}_{\rightarrow} + \mathcal{U}_{\leftarrow} = \mathcal{U}_+ + \mathcal{U}_-$.*

Proof. Let $u = \sum_{n=-\infty}^{\infty} a_n q^n$ be either an element of $R((q))\widehat{\ }$ or of $R((q^{-1}))\widehat{\ }$. We express the $\{\delta_p, \delta_q\}$-character ψ_a as $\psi_a = \mu(\phi_p, \delta_q)y$, for some polynomial $\mu(\xi_p, \xi_q) = \sum_{i,j \geq 0} \mu_{ij} \xi_p^i \xi_q^j \in R[\xi_p, \xi_q]$. Then $\psi_a u = 0$ if, and only if,

$$\mu(0, n)a_n + \sum_{j \geq 0} \sum_{i \geq 1} \mu_{ij} (n/p^i)^j a_{n/p^i}^{\phi^i} = 0. \tag{73}$$

for all $n \in \mathbb{Z}$. In this last expression, $a_{n/p^i} = 0$ if $n/p^i \notin \mathbb{Z}$. Thus, if $\mu(0, n) \neq 0$ for all $n \in \mathbb{Z}$, we derive by induction that $a_n = 0$ for all $n \neq 0$, which proves the first part.

In order to prove 2), we first note that, by Lemma 6.3, $u_{a,\kappa,\alpha} \in \mathcal{U}_{\pm 1}$ according as $\kappa \in \mathcal{K}_{\pm}$ respectively. Now if $u^* \in \mathcal{U}_1$, that is to say, if $u^* = \sum_{n \geq 1} a_n q^n$, we set

$$u^{**} := u^* - \sum_{\kappa \in \mathcal{K}_+} u_{a,\kappa,a_\kappa} \in \mathcal{U}_1 \, .$$

Set $\rho_+ := \sharp \mathcal{K}_+$. Using (73), one easily checks that the map $B_+^0 : \mathcal{U}_+ \to R^{\rho_+}$ defined by

$$B_+^0 (\sum a_n q^n) = (a_\kappa)_{\kappa \in \mathcal{K}_+}$$

is injective. On the other hand, by (69), we have $B_+^0 u^{**} = 0$. Thus, $u^{**} = 0$, and $u^* = \sum u_{a,\kappa.a_\kappa}$. A similar argument holds for $\mathcal{U}_{-1}$. This completes the proof of the second part.

In order to prove 3), let us note that if

$$u = \sum_{n=-\infty}^{\infty} a_n q^n \in \mathcal{U}_{\rightarrow} \, ,$$

it is then clear that

$$\psi_a\left(\sum_{n<0} a_n q^n\right) = 0\,, \text{ and } \psi_a\left(\sum_{n\geq 0} a_n q^n\right) = 0\,.$$

Thus, $\sum_{n<0} a_n q^n \in \mathcal{U}_-$ and $\sum_{n\geq 0} a_n q^n \in \mathcal{U}_+$. Therefore, $u \in \mathcal{U}_- + \mathcal{U}_+$, and so $\mathcal{U}_\rightarrow \subset \mathcal{U}_- + \mathcal{U}_+$. A similar argument shows that $\mathcal{U}_\leftarrow \subset \mathcal{U}_- + \mathcal{U}_+$. □

Example 6.11. Let us examine a special case of Theorem 6.10. For integers $r, s \geq 1$ and $\lambda \in R^\times$, we consider the $\{\delta_p, \delta_q\}$-character

$$\psi_a := (\delta_q^r + \lambda\phi_p^s - \lambda)y\,, \tag{74}$$

If (r, s) is any one of the pairs $(1,1), (1,2), (2,1), (2,2)$, then ψ_a can be viewed as an analogue of the convection equation, heat equation, sideways heat equation, or wave equation, respectively. The characteristic polynomial of ψ_a is

$$\mu(\xi_p, \xi_q) = \xi_q^r + \lambda\xi_p^s - \lambda\,.$$

Clearly μ is unmixed. The characteristic integers are the integer roots of the equation

$$\xi_q^r - \lambda = 0\,.$$

Thus, if $\lambda \notin \{n^r \mid n \in \mathbb{Z}\}$, there are no characteristic integers, and $\mathcal{U}_\rightarrow = \mathcal{U}_\leftarrow = \mathcal{U}_0 = R^{\phi^s}$.

Assume in what follows that $\lambda = n^r$ for some $n \in \mathbb{Z}$. For even r, we may assume further that $n > 0$. Then $\mathcal{K} = \{n\}$ for r odd, and $\mathcal{K} = \{-n, n\}$ for r even. The basic series for $\kappa \in \mathcal{K}$ are

$$u_{a,\kappa,\alpha} = \sum_{j\geq 0} (-1)^j \frac{1}{F_j(p^{sr})} \phi_p^{sj}(\alpha q^\kappa)\,, \tag{75}$$

where $F_j(x) \in \mathbb{Z}[x]$ are the polynomials $F_0(x) = 1$,

$$F_j(x) := \prod_{i=1}^{j} (x^i - 1)\,,\ j \geq 1\,.$$

Notice that the integers $F_j(p^{sr})$ have a nice simple interpretation in terms of flags:

$$F_j(p^{sr}) = (p^{sr} - 1)^j \cdot \sharp(GL_j(\mathbb{F}_{p^{sr}})/B_j(\mathbb{F}_{p^{sr}}))\,,$$

where $B_j(\mathbb{F}_{p^{sr}})$ is the subgroup of $GL_j(\mathbb{F}_{p^{sr}})$ consisting of all upper triangular matrices. Also notice that, for $\kappa \in \mathcal{K}$, we have that $\bar{u}_{a,\kappa,\alpha} \in k[[q^{\pm 1}]]$, the reduction mod p of $u_{a,\kappa,\alpha}$, is given by

$$\bar{u}_{a,\kappa,\alpha} = \sum_{n\geq 0} \bar{\alpha}^{p^n} q^{\kappa p^n}\,,$$

so $\bar{u}_{a,\kappa,\alpha}$ is a root of the Artin-Schreier polynomial

$$t^p - t + \bar{\alpha}q^\kappa \in k(q)[t]\,.$$

We have the following:

a) For $n > 0$ and odd r,

$$\begin{array}{rcl} \mathcal{U}_{-1} & = & 0\,, \\ \mathcal{U}_1 & = & \{u_{a,n,\alpha} \mid \alpha \in R\}\,. \end{array}$$

b) For $n<0$ and odd r,

$$\begin{array}{rcl} \mathcal{U}_{-1} & = & \{u_{a,n,\alpha} \mid \alpha \in R\}\,, \\ \mathcal{U}_{1} & = & 0\,. \end{array}$$

c) For r even,

$$\begin{array}{rcl} \mathcal{U}_{-1} & = & \{u_{a,-n,\alpha} \mid \alpha \in R\}\,, \\ \mathcal{U}_{1} & = & \{u_{a,n,\alpha} \mid \alpha \in R\}\,. \end{array}$$

d) $\mathcal{U}_{\rightarrow} = \mathcal{U}_{+}, \quad \mathcal{U}_{\leftarrow} = \mathcal{U}_{-}$.

Indeed, (a), (b) and (c) follow directly by Theorem 6.10. The two families of solutions in (c) should be viewed as analogues of the two waves traveling in opposite directions in the case of the classical wave equation. In contrast to this, we have only one "wave" in (a), which is the case of the "convection" equation.

We prove the first equality in (d). The second follows by a similar argument. Let

$$u = \sum_{n=-\infty}^{\infty} a_n q^n \in \mathcal{U}_{\rightarrow}\,.$$

It is clear that

$$\psi_a\left(\sum_{n<0} a_n q^n\right) = 0\,.$$

By (a), (b) and (c), we must have that

$$\sum_{n<0} a_n q^n = u_{a,-|\kappa|,\alpha}$$

for some $\alpha \in R$. But $a_n \to 0$ as $n \to -\infty$, and this is the case for $u_{a,-|\kappa|,\alpha}$ only when $\alpha = 0$ (see (75)). Thus, $\alpha = 0$, and $u \in \mathcal{U}_{+}$. □

We derive here some consequences of Theorem 6.10.

Corollary 6.12. *Under the hypotheses of Theorem* 6.10 *let* $u \in \mathcal{U}_{\pm 1}$. *Then the following hold:*

1. *The series* $\bar{u} \in k[[q^{\pm 1}]]$ *is integral over* $k[q^{\pm 1}]$ *and the field extension* $k(q) \subset k(q,\bar{u})$ *is Abelian with Galois group killed by* p.
2. *If the characteristic polynomial of* ψ_a *is unmixed and* $\mathcal{K}_{\pm}$ *is short then* u *is transcendental over* $K(q)$.

Proof. This follows directly from Theorem 6.10 and Lemma 6.6. □

Corollary 6.13. *Under the hypotheses of Theorem* 6.10 *the maps* $B^0_{\pm} : \mathcal{U}_{\pm 1} \to R^{\rho_{\pm}}$ *are* R*-module isomorphisms. Furthermore, for any* $u \in \mathcal{U}_{\pm 1}$ *we have*

$$u = \sum_{\kappa \in \mathcal{K}_{\pm}} (B^0_{\kappa} u) \star u_{a,\kappa,1}\,.$$

In particular the "boundary value problem at $q^{\pm 1} = 0$" is well posed.

The next Corollary says that the "boundary value problem at $q \neq 0$" is well posed.

Corollary 6.14. *Under the hypotheses of Theorem* 6.10, *assume further that* $\mathcal{K}_{+} = \{\kappa\}$. *Then for any* $q_0 \in p^{\nu} R^{\times}$ *with* $\nu \geq 1$, *and any* $g \in p^{\kappa\nu} R$, *there exists a unique* $u \in \mathcal{U}_1$ *such that* $u(q_0) = g$.

Proof. We need to show that the map

$$\begin{array}{rcl} R & \rightarrow & p^{\kappa\nu}R \\ \alpha & \mapsto & \sum_{n\geq 0} b_{n,\kappa}\alpha^{\phi^n}q_0^{\kappa p^n} \end{array}$$

is bijective. This follows by Lemma 6.15 below. □

Lemma 6.15. *Let $c_0 \in R^\times$, $c_1, c_2, c_3, \ldots \in pR$ and $c_n \to 0$ p-adically as $n \to \infty$. Then the map*

$$\begin{array}{rcl} R & \rightarrow & R \\ \alpha & \mapsto & \sum_{n\geq 0} c_n\alpha^{\phi^n} \end{array}$$

is bijective.

Proof. The injectivity is clear. And surjectivity follows by a Hensel-type argument. □

The following Corollary is concerned with the inhomogeneous equation $\psi_a u = \varphi$. Recall that for $\varphi \in q^{\pm 1}R[[q^{\pm 1}]]$ we define the *support* of φ as the set $\{n; c_n \neq 0\}$. This notion of support is standard for series but note that it is not a direct analogue of the notion of support in real analysis if one pursues the analogy according to which q is an analogue of the exponential of complex time. Also, for any series $v \in R((q^{\pm 1}))\widehat{\ }$ we denote by $\bar{v} \in k((q^{\pm 1}))$ the reduction of v mod p.

Corollary 6.16. *Let ψ_a be a non-degenerate $\{\delta_p, \delta_q\}$-character of $\mathbb{G}_a$ and let $\varphi \in q^{\pm 1}R[[q^{\pm 1}]]$ be a series whose support is contained in the set $\mathcal{K}'$ of totally non-characteristic integers of ψ_a. Then the following hold:*

1. *The equation $\psi_a u = \varphi$ has a unique solution $u \in \mathbb{G}_a(q^{\pm 1}R[[q^{\pm 1}]])$ such that u has support disjoint from the set $\mathcal{K}$ of characteristic integers.*
2. *If $\bar{\varphi} \in k[q^{\pm 1}]$ then $\bar{u} \in k[[q^{\pm 1}]]$ is integral over $k[q^{\pm 1}]$ and the field extension $k(q) \subset k(q, \bar{u})$ is Abelian with Galois group killed by p.*
3. *If the characteristic polynomial of ψ_a is unmixed and the support of φ is short then u is transcendental over $K(q)$.*

Proof. The existence in assertion 1 follows from Lemma 6.3. Uniqueness follows from Lemma 6.13. Assertions 2 and 3 follows from 6.6. □

Remark 6.17. Corollary 6.14 implies that if ψ_a is non-degenerate and $\mathcal{K}_+ = \{1\}$, for any $q_0 \in pR^\times$ the group homomorphism

$$\begin{array}{rcl} R & \stackrel{S_{q_0}}{\rightarrow} & \mathbb{G}_a(R) = R \\ \alpha & \mapsto & \frac{1}{p}u_{a,1,\alpha}(q_0) \end{array}$$

is an isomorphism. Thus, for any $q_1, q_2 \in pR^\times$, we have an isomorphism

$$S_{q_1,q_2} := S_{q_2} \circ S_{q_1}^{-1} : \mathbb{G}_a(R) \to \mathbb{G}_a(R)\,.$$

mapping that can be viewed as the "propagator" attached to ψ_a. Note that if $\zeta \in \mu(R)$ and $q_0 \in pR^\times$, then by (71) we have that

$$S_{\zeta q_0}(\alpha) = \frac{1}{p}u_{a,1,\alpha}(\zeta q_0) = \frac{1}{p}u_{a,1,\zeta\alpha}(q_0) = S_{q_0}(\zeta\alpha)\,,$$

so

$$S_{\zeta q_0} = S_{q_0} \circ M_\zeta\,,$$

where $M_\zeta : R \to R$ is the mapping defined by $M_\zeta(\alpha) := \zeta\alpha$. Thus, for $\zeta_1, \zeta_2 \in \mu(R)$, we get that

$$S_{\zeta_1 q_0, \zeta_2 q_0} = S_{q_0} \circ M_{\zeta_2/\zeta_1} \circ S_{q_0}^{-1} \, .$$

In particular,

$$S_{q_0, \zeta_1\zeta_2 q_0} = S_{q_0, \zeta_2 q_0} \circ S_{q_0, \zeta_1 q_0} \, .$$

This latter equality can be interpreted as a (weak) incarnation of "Huygens principle" ([25], p. 104).

7. Multiplicative group

In this section we prove our main results about $\{\delta_p, \delta_q\}$-characters and their space of solutions in the case where G is the multiplicative group.

Let $\mathbb{G}_m := \mathrm{Spec}\, A[y, y^{-1}]$ be the multiplicative group over our fixed $\{\delta_p, \delta_q\}$-ring A. We equip $\mathbb{G}_m$ with the invariant 1-form

$$\omega := \frac{dy}{y} \, .$$

Let us consider the $\{\delta_p, \delta_q\}$-characters

$$\psi_p, \psi_q \in \mathbf{X}^1_{pq}(\mathbb{G}_m) \subset A[y, y^{-1}, \delta_p y, \delta_q y]\widehat{\ }$$

defined by

$$\begin{aligned} \psi_p = \psi_{m,p} \quad &:= \quad \frac{1}{p}\log\left(\frac{\phi_p(y)}{y^p}\right) = \frac{1}{p}\log\left(1 + p\frac{\delta_p y}{y^p}\right) = \frac{\delta_p y}{y^p} - \frac{p}{2}\left(\frac{\delta_p y}{y^p}\right)^2 + \cdots , \\ \psi_q = \psi_{m,q} \quad &:= \quad \delta_q \log y := \frac{\delta_q y}{y} \, . \end{aligned}$$

Here, if $y = 1 + T$, then

$$\log y := l(T) = T - \frac{T^2}{2} + \frac{T^3}{3} - \cdots$$

is the logarithm of the formal group of $\mathbb{G}_m$. We clearly have $\psi_p \in \mathbf{X}^1_p(\mathbb{G}_m)$, and $\psi_q \in \mathbf{X}^1_q(\mathbb{G}_m)$. The images of ψ_p and ψ_q in $A[[T]][\delta_p T, \delta_q T]\widehat{\ }$ are

$$\begin{aligned} \psi_p &= \frac{1}{p}(\phi_p - p) l(T) \, , \\ \psi_q &= \delta_q l(T) \, . \end{aligned}$$

Lemma 7.1. *We have that $\delta_q \psi_p = (\phi_p - 1)\psi_q$ in $\mathbf{X}^2_{pq}(\mathbb{G}_m)$.*

Proof. By a direct calculation,

$$\delta_q \psi_p = \delta_q \left(\frac{1}{p}\left(\phi_p - p\right) l(T)\right) = (\phi_p - 1)\delta_q l(T) = (\phi_p - 1)\psi_q \, .$$

∎

Proposition 7.2. *For each $r \geq 1$, the L-vector space $\mathbf{X}^r_{pq}(\mathbb{G}_m) \otimes_A L$ has basis*

$$\{\phi_p^i \psi_p \mid_{0 \leq i \leq r-1}\} \cup \{\phi_p^i \delta_q^j \psi_q \mid_{0 \leq i+j \leq r-1}\} \, .$$

In particular, ψ_p and ψ_q generate the $L[\phi_p, \delta_q]$-module $\mathbf{X}^\infty_{pq}(\mathbb{G}_m) \otimes L$.

Proof. There is an exact sequence of homomorphisms of groups in the category of p-adic formal schemes

$$0 = \mathrm{Hom}(\hat{\mathbb{G}}_m, \hat{\mathbb{G}}_a) \stackrel{\pi_r^*}{\to} \mathrm{Hom}(J^r_{pq}(\mathbb{G}_m), \hat{\mathbb{G}}_a) \stackrel{\rho}{\to} \mathrm{Hom}(N^r, \hat{\mathbb{G}}_a)\,,$$

where $\pi_r : J^r_{pq}(\mathbb{G}_m) \to \hat{\mathbb{G}}_a$ is the natural projection, $N^r = \ker \pi_r$, and ρ is defined by restriction. We recall that $\mathrm{Hom}(J^r_{pq}(\mathbb{G}_m), \hat{\mathbb{G}}_a)$ identifies with the module of $\{\delta_p, \delta_q\}$-characters $\mathbf{X}^r_{pq}(\mathbb{G}_m)$. Looking at the level of Lie algebras, we see that the rank of $\mathrm{Hom}(N^r, \hat{\mathbb{G}}_a)$ over A is at most equal to its dimension, $r(r+3)/2$. Since ρ is injective, it is enough to show that the family in the statement of the Proposition is A-linearly independent. Thus, it is enough to show that the image of this family via the map (42) is A-linearly independent. But

$$\begin{array}{rcl}(\phi_p^i \psi_p) \circ e(pT) & = & \phi_p^i \left(\dfrac{1}{p}(\phi_p - p) l(e(pT))\right) = \phi_p^{i+1} T - p\phi_p^i T\,, \\ (\phi_p^i \delta_q^j \psi_q) \circ e(pT) & = & \phi_p^i \delta_q^{j+1} l(e(pT)) = p\phi_p^i \delta_q^{j+1} T\,,\end{array}$$

and it is rather clear that these elements are A-linearly independent. $\square$

From now on, we let $A = R$, hence $L = K$, and we use the notation and discussion in Example 2.16 applied to $\mathbb{G}_m$ over R. We have a natural embedding

$$\begin{array}{rcl}\iota : q^{\pm 1} R[[q^{\pm 1}]] & \to & \mathbb{G}_m(q^{\pm 1} R[[q^{\pm 1}]]) \\ u & \mapsto & \iota(u) = 1 + u\end{array}$$

By Proposition 7.2, any $\{\delta_p, \delta_q\}$-character of $\mathbb{G}_m$ is a K-multiple of a $\{\delta_p, \delta_q\}$-character of the form

$$\psi_m := \nu(\phi_p, \delta_q)\psi_q + \lambda(\phi_p)\psi_p\,, \tag{76}$$

where $\nu(\xi_p, \xi_q) \in R[\xi_p, \xi_q]$, $\lambda(\xi_p) \in R[\xi_p]$. The Picard-Fuchs symbol of ψ_m with respect to the étale coordinate $T = y - 1$ is trivially seen to be

$$\sigma(\xi_p, \xi_q) = p\nu(\xi_p, \xi_q)\xi_q + \lambda(\xi_p)(\xi_p - p).$$

Hence, the Fréchet symbol with respect to the invariant form $\omega = \frac{dy}{y} = \frac{dT}{1+T}$ is given by

$$\theta(\xi_p, \xi_q) = \frac{\sigma(p\xi_p, \xi_q)}{p} = \nu(p\xi_p, \xi_q)\xi_q + \lambda(p\xi_p)(\xi_p - 1)\,.$$

Definition 7.3. Let ψ_m be a $\{\delta_p, \delta_q\}$-character of $\mathbb{G}_m$ of the form (76). We define the *characteristic polynomial* $\mu(\xi_p, \xi_q)$ of ψ_m to be the Fréchet symbol $\theta(\xi_p, \xi_q)$ of ψ_m with respect to ω. We say that the $\{\delta_p, \delta_q\}$-character ψ_m is *non-degenerate* if $\mu(0,0) \in R^\times$, or equivalently, if $\lambda(0) \in R^\times$. For a non-degenerate character ψ_m with symbol $\mu(\xi_p, \xi_q)$, we define the *characteristic integers* to be the integers κ that are solutions of the equation $\mu(0, \kappa) = 0$. Any such characteristic integer κ must be coprime to p. We say that $\kappa \in \mathbb{Z}$ is *totally non-characteristic* if $\kappa \not\equiv 0 \bmod p$ and $\mu(0, \kappa) \not\equiv 0 \bmod p$. We denote by $\mathcal{K}$ the set of all characteristic integers and set $\mathcal{K}_\pm := \mathcal{K} \cap \mathbb{Z}_\pm$. Also we denote by $\mathcal{K}'$ the set of totally non-characteristic integers. For any $0 \neq \kappa \in \mathbb{Z}$ and $\alpha \in R$, we define the *basic series*

$$u_{m,\kappa,\alpha} := \exp\left(\int u^\mu_{a,\kappa,\alpha} \frac{dq}{q}\right) \in 1 + \alpha \frac{q^\kappa}{\kappa} + \cdots \in K[[q^{\pm 1}]]\,, \tag{77}$$

where $u^{\mu}_{a,\kappa,\alpha}$ is as in (66),

$$\exp(T) = 1 + T + \frac{T^2}{2!} + \frac{T^3}{3!} + \cdots ,$$

and

$$v \mapsto \int v dq$$

is the usual indefinite integration $K[[q]] \to qK[[q]]$ or the integration $q^{-2}K[[q^{-1}]] \to q^{-1}K[[q^{-1}]]$, according to the cases κ positive or negative, respectively.

Example 7.4. Consider the $\{\delta_p, \delta_q\}$-characters

$$\psi_m := \delta_q^{r-1}\psi_q + (\lambda_{s-1}\phi_p^{s-1} + \cdots + \lambda_1\phi_p + \lambda_0)\psi_p ,$$

where $\lambda_0 \in R^{\times}$, $\lambda_1, \ldots, \lambda_{s-1} \in R$. If (r, s) is any one of the pairs $(1, 1)$, $(1, 2)$, $(2, 1)$, $(2, 2)$, then ψ_m can be viewed as an analogue of the convection equation, heat equation, sideways heat equation, or wave equation respectively. The characteristic polynomial of ψ_m is

$$\mu(\xi_p, \xi_q) = \xi_q^r + (p^{s-1}\lambda_{s-1}\xi_p^{s-1} + \cdots + p\lambda_1\xi_p + \lambda_0)(\xi_p - 1) ,$$

hence μ is unmixed, and the characteristic integers are the integer roots of the equation

$$\xi_q^r - \lambda_0 = 0 .$$

In particular, for

$$\psi_m := \delta_q^{r-1}\psi_q + \lambda\psi_p , \tag{78}$$

the characteristic polynomial equals

$$\mu(\xi_p, \xi_q) = \xi_q^r + \lambda\xi_p - \lambda ,$$

which is the case $s = 1$ of the Example 6.11. So in this case $u^{\mu}_{a,\kappa,\alpha}$ in (77) is given by

$$u^{\mu}_{a,\kappa,\alpha} = \sum_{n\geq 0}(-1)^n \frac{1}{F_n(p^r)}\phi_p^n(\alpha q^{\kappa}) . \tag{79}$$

Example 7.5. Let us consider the "simplest" possible energy function

$$H = a(\psi_q)^2 + 2b\psi_p\psi_q + c(\psi_p)^2 ,$$

for $a, b, c \in R$. Using the obvious equalities

$$\begin{array}{rcl} \theta_{\psi_q,\omega} & = & \xi_q , \\ \theta_{\psi_p,\omega} & = & \xi_p - 1 , \end{array}$$

and (64), we obtain the following formula for the Euler-Lagrange equation associated to H and the vector field $\partial = y\partial_y$:

$$\epsilon^1_{H,\partial} = (-2a^{\phi}\phi_p\delta_q - 2b^{\phi}\phi_p^2 + 2b)\psi_q + (-2c^{\phi}\phi_p + 2c)\psi_p .$$

This $\{\delta_p, \delta_q\}$-character $\epsilon^1_{H,\partial}$ is non-degenerate if, and only if, $c \in R^{\times}$, and its characteristic polynomial is given by

$$\mu(\xi_p, \xi_q) = (-2b^{\phi}p^2\xi_p^2 - 2a^{\phi}p\xi_p\xi_q + 2b)\xi_q + (-2c^{\phi}p\xi_p + 2c)(\xi_p - 1) .$$

The set of characteristic integers of $\epsilon^1_{H,\partial}$ is

$$\mathcal{K} = \left\{\frac{c}{b}\right\} \cap \mathbb{Z} .$$

Lemma 7.6. *Let $\kappa \in \mathbb{Z}\backslash p\mathbb{Z}$. Then, for any $\alpha \in R$, we have that $u_{m,\kappa,\alpha} \in 1 + q^{\pm 1}R[[q^{\pm 1}]]$. Furthermore, the mapping*

$$\begin{array}{rcl} R & \rightarrow & R[[q^{\pm 1}]]^{\times} = \mathbb{G}_m(R[[q^{\pm 1}]]) \\ \alpha & \mapsto & u_{m,\kappa,\alpha} \end{array}$$

is a pseudo δ_p-polynomial map.

Proof. We let $u := u_{m,\kappa,\alpha} = \exp h$, where $h := \int u^{\mu}_{a,\kappa,\alpha}\frac{dq}{q}$. By Dwork's Lemma 2.4, in order to show that $u_{m,\kappa,\alpha} \in 1 + q^{\pm 1}R[[q^{\pm 1}]]$ it is enough to prove that $u^{\phi}/u^p \in 1 + pq^{\pm 1}R[[q^{\pm 1}]]$. Since $u^{\phi}/u^p = \exp((\phi - p)h)$, we just need to show that

$$(\phi_p - p)h \in pq^{\pm 1}R[[q^{\pm 1}]]\,. \tag{80}$$

Let us start by observing that

$$\delta_q h = u^{\mu}_{a,\kappa,\alpha}\,, \tag{81}$$

and consider the $\{\delta_p, \delta_q\}$-character of $\mathbb{G}_a$ defined by

$$\psi_a := \psi^{\mu}_a := \mu(\phi_p, \delta_q)y\,.$$

By Lemma 6.3, we obtain that

$$\begin{array}{rcl} \delta_q(p\mu(0,\kappa)\alpha\kappa^{-1}q^{\kappa}) & = & p\mu(0,\kappa)\alpha q^{\kappa} \\ & = & p\psi^{\mu}_a u^{\mu}_{a,\kappa,\alpha} \\ & = & p[\nu(p\phi_p, \delta_q)\delta_q + \lambda(p\phi_p)(\phi_p - 1)]\delta_q h \\ & = & \delta_q[p\nu(\phi_p, \delta_q)\delta_q + \lambda(\phi_p)(\phi_p - p)]h\,. \end{array}$$

We deduce that

$$[p\nu(\phi_p, \delta_q)\delta_q + \lambda(\phi_p)(\phi_p - p)]h - p\mu(0,\kappa)\alpha\kappa^{-1}q^{\kappa} \in q^{\pm 1}K[[q^{\pm 1}]] \cap K = 0\,,$$

and consequently,

$$\begin{array}{rcl} \lambda(\phi_p)(\phi_p - p)h & = & -p\nu(\phi_p, \delta_q)\delta_q h + p\mu(0,\kappa)\alpha\kappa^{-1}q^{\kappa} \\ & = & -p\nu(\phi_p, \delta_q)u^{\mu}_{a,\kappa,\alpha} + p\mu(0,\kappa)\alpha\kappa^{-1}q^{\kappa} \\ & \in & pq^{\pm 1}R[[q^{\pm 1}]]\,. \end{array}$$

By Lemma 2.6, (80) then follows.

In order to prove the remaining assertion, let us assume that $\kappa > 0$. The opposite case can be argued similarly. Note that we can find polynomials $Q_n \in R[z_0, z_1, \ldots, z_{r_n}]$, and integers $m_n \geq 0$, such that

$$u_{m,\kappa,\alpha} = \sum_{n \geq 1} p^{-m_n} Q_n(\alpha, \alpha^{\phi}, \ldots, \alpha^{\phi^{r_n}})q^n$$

for all $\alpha \in R$. By the part of the Lemma already proven, we have

$$p^{-m_n} Q_n(\alpha, \alpha^{\phi}, \ldots, \alpha^{\phi^{r_n}}) \in R$$

for all $\alpha \in R$. We then apply Corollary 3.21 in [9], and conclude that there exists a polynomial $P_n \in R[x_0, x_1, \ldots, x_{r_n}]$ such that

$$p^{-m_n} Q_n(\alpha, \alpha^{\phi}, \ldots, \alpha^{\phi^{r_n}}) = P_n(\alpha, \alpha^{\phi}, \ldots, \alpha^{\phi^{r_n}})$$

for all $\alpha \in R$. Thus, the mapping

$$\begin{array}{rcl} R & \rightarrow & R[[q]] \\ \alpha & \mapsto & u_{m,\kappa,\alpha} \end{array}$$

is pseudo δ_p-polynomial. Then

$$\begin{array}{ccc} R & \rightarrow & R[[q]] \\ \alpha & \mapsto & \frac{1}{u_{m,\kappa,\alpha}} \end{array}$$

is pseudo δ_p-polynomial also. By considering the embedding

$$\begin{array}{ccc} \mathbb{G}_m & \rightarrow & \mathbb{A}^2 \\ x & \mapsto & (x, x^{-1}) \end{array} ,$$

the last assertion of the Lemma follows. $\square$

As in the additive case, we have the following "diagonalization" result.

Lemma 7.7. *For all $\kappa \in \mathbb{Z}\backslash p\mathbb{Z}$ and $\alpha \in R$, we have that*

$$\begin{array}{rcl} \psi_q u_{m,\kappa,\alpha} & = & u^{\mu}_{a,\kappa,\alpha} \\ \psi_m u_{m,\kappa,\alpha} & = & \mu(0,\kappa) \cdot \alpha\kappa^{-1} q^{\kappa} . \end{array}$$

Proof. The first equality is clear. Let us assume now that $\kappa > 0$. A similar argument can be used to handle the case $\kappa < 0$. By Lemma 7.1, we obtain that

$$\begin{array}{rcl} \delta_q \psi_m & = & \delta_q \left[\nu(\phi_p, \delta_q)\psi_q + \lambda(\phi_p)\psi_p \right] \\ & = & [\nu(p\phi_p, \delta_q)\delta_q + \lambda(p\phi_p)(\phi_p - 1)]\psi_q \\ & = & \psi^{\mu}_a \psi_q , \end{array}$$

where $\psi^{\mu}_a := \mu(\phi_p, \delta_q) y$. In particular, by Lemma 6.3 we have that

$$\begin{array}{rcl} \delta_q \psi_m u_{m,\kappa,\alpha} & = & \psi^{\mu}_a \psi_q u_{m,\kappa,\alpha} \\ & = & \psi^{\mu}_a u^{\mu}_{a,\kappa,\alpha} \\ & = & \mu(0,\kappa)\alpha q^{\kappa} \\ & = & \delta_q(\mu(0,\kappa)\alpha\kappa^{-1} q^{\kappa}) , \end{array}$$

and so $v := \psi_m u_{m,\kappa,\alpha} - \mu(0,\kappa)\alpha\kappa^{-1}q^{\kappa} \in R$. On the other hand,

$$v(0) = (\psi_m u_{m,\kappa,\alpha})(0) = \psi_m(u_{m,\kappa,\alpha}(0)) = \psi_m(1) = 0 ,$$

hence $v = 0$, and we are done. $\square$

Remark 7.8.

a) For all $\kappa \in \mathbb{Z}\backslash p\mathbb{Z}$ the map

$$\begin{array}{ccc} R & \rightarrow & R[[q^{\pm 1}]]^{\times} \\ \alpha & \mapsto & u_{m,\kappa,\alpha} \end{array}$$

is an injective homomorphism.

b) We recall (see (31)) the natural group homomorphisms attached to ψ_q,

$$\begin{array}{ccc} B^0_{\kappa} : R[[q^{\pm 1}]]^{\times} & \rightarrow & R \\ B^0_{\kappa} u & = & \Gamma_{\kappa}\psi_q u, \end{array}$$

where $\kappa \in \mathbb{Z}\backslash p\mathbb{Z}$ and the homomorphism

$$\begin{array}{ccc} B^0_{\pm} : R[[q^{\pm 1}]]^{\times} & \rightarrow & R^{\rho_{\pm}} \\ B^0_{\pm} u & = & (\Gamma_{\kappa}\psi_q u)_{\kappa \in \mathcal{K}_{\pm}} . \end{array}$$

For integers $\kappa_1, \kappa_2 \in \mathbb{Z}\backslash p\mathbb{Z}$ we get that

$$B^0_{\kappa_1} u_{m,\kappa_2,\alpha} = \Gamma_{\kappa_1} u^{\mu}_{a,\kappa_2,\alpha} = \alpha \cdot \delta_{\kappa_1 \kappa_2} . \tag{82}$$

c) For $\kappa \in \mathbb{Z}\backslash p\mathbb{Z}$ we see that

$$u_{m,\kappa,\zeta^\kappa\alpha}(q) = u_{m,\kappa,\alpha}(\zeta q) \tag{83}$$

for all $\zeta \in \mu(R)$. In particular, if $\alpha = \sum_{i=0}^\infty m_i\zeta_i^\kappa$, $\zeta_i \in \mu(R)$, $m_i \in \mathbb{Z}$, $v_p(m_i) \to \infty$, we have

$$u_{m,\kappa,\alpha}(q) = \prod_{i=0}^{\infty}(u_{m,\kappa,1}(\zeta_i q))^{m_i}\, .$$

(Note that the right hand side of the equality above converges in the $(p, q^{\pm 1})$-adic topology of $R[[q^{\pm 1}]]$.) Thus, if $f \in \mathbb{Z}\mu(R)\widehat{\ }$ is such that $(f^{[\kappa]})^\sharp = \alpha \in R$, then $u_{m,\kappa,\alpha}$ can be expressed via convolution by

$$u_{m,\kappa,\alpha} = f \star u_{m,\kappa,1}\, .$$

Under convolution, the set $\{u_{m,\kappa,\alpha} \mid \alpha \in R\}$ is a $\mathbb{Z}\mu(R)\widehat{\ }$-module, whose module structure arises from an R-module structure (still denoted by $\star$) induced by base change via the morphism

$$\mathbb{Z}\mu(R)\widehat{\ } \stackrel{[\kappa]}{\to} \mathbb{Z}\mu(R)\widehat{\ } \stackrel{\sharp}{\to} R\, .$$

(Cf. to (34)). The R-module $\{u_{m,\kappa,\alpha} \mid \alpha \in R\}$ is free with basis $u_{m,\kappa,1}$. So, for $g \in \mathbb{Z}\mu(R)\widehat{\ }$, $\beta = (g^{[\kappa]})^\sharp$, we have that

$$\beta \star u_{m,\kappa,\alpha} = g \star u_{m,\kappa,\alpha}\, ,$$

and in particular

$$u_{m,\kappa,\alpha} = \alpha \star u_{m,\kappa,1}\, .$$

d) We have the following "rationality" property: if $\alpha \in \mathbb{Z}_{(p)}$, then $u_{m,\kappa,\alpha} \in \mathbb{Z}_{(p)}[[q^{\pm 1}]]$ for $\kappa \in \mathbb{Z}\backslash p\mathbb{Z}$.

Theorem 7.9. *Let ψ_m be a non-degenerate $\{\delta_p, \delta_q\}$-character of $\mathbb{G}_m$, $\mathcal{U}_*$ the corresponding groups of solutions, $\mathcal{K}$ the set of characteristic integers, and $u_{m,\kappa,\alpha}$ the basic series. We set*

$$\begin{array}{rcll} \mathcal{U}_{tors} & := & \mu(R) & \subset\ \mathcal{U}_0 \\ \mathcal{U}_\sim & := & (R^\times \cdot q^{\mathbb{Z}}) \cap \mathcal{U} & \subset\ \mathcal{U}_\downarrow\, . \end{array}$$

Then the following hold:

1) $\mathcal{U}_\to = \mathcal{U}_\sim \cdot \mathcal{U}_1$, *and* $\mathcal{U}_\leftarrow = \mathcal{U}_\sim \cdot \mathcal{U}_{-1}$.
2) *We have that*

$$\mathcal{U}_{\pm 1} \quad = \quad \prod_{\kappa \in \mathcal{K}_\pm} \{u_{m,\kappa,\alpha} \mid \alpha \in R\}\, ,$$

where $\prod$ stands for internal direct product. In particular, $\mathcal{U}_{\pm 1}$ are free R-modules under convolution, with bases $\{u_{m,\kappa,1} \mid \kappa \in \mathcal{K}_\pm\}$ respectively.

3) *We have that* $\mathcal{U}_0 = \mathcal{U}_{tors}$.

Remark 7.10. The elements $u = bq^n$ of $R^\times \cdot q^{\mathbb{Z}}$ deserve to be called *plane waves* with

frequency	$n \in \mathbb{Z}\,,$
wave number	$\gamma := \psi_{m,p}(b) = \dfrac{1}{p}\log\dfrac{\phi_p(b)}{b^p} \in R\,,$
wave length	$1/\gamma \in K \cup \{\infty\}\,,$
propagation speed	$n/\gamma \in K \cup \{\infty\}\,.$

This can be justified by the analogy behind the chosen terminology. Indeed, if $u = bq^n$, then

$$n = \frac{\delta_q u}{u}\,,$$

and so n is to be interpreted as the "logarithmic derivative" of u with respect to δ_q. Also, we have that

$$\gamma = \frac{1}{p}\log\frac{\phi_p(u)}{u^p}\,,$$

and so γ is the analogue of a "logarithmic derivative of u with respect to δ_p." These observations suffice to justify the assertion.

For in the classical theory, the complex valued function $u(t,x)$ of real variables t and x defined by

$$u(t,x) := a(x)e^{-2\pi i\nu t}$$

is viewed as a plane wave with frequency ν, wave number (at x) $\gamma = \gamma(x) = \frac{1}{2\pi i}\frac{a'(x)}{a(x)}$, wave length $1/\gamma$, and propagation speed ν/γ. (The standard situation is that in which $a(x) = e^{2\pi i\gamma x}$, where $\gamma \neq 0$ is a constant.) Notice that these ν and γ are equal to

$$-(2\pi i)^{-1}\frac{\partial_t u}{u} \quad \text{and} \quad (2\pi i)^{-1}\frac{\partial_x u}{u}\,,$$

respectively. Thus, our terminology correspond to these classical definitions of frequency and wave number provided that δ_q and δ_p are viewed as analogues of $-(2\pi i)^{-1}\partial_t$ and $(2\pi i)^{-1}\partial_x$, respectively.

The solutions in $\mathcal{U}_\sim$ correspond to those solutions that in the classical case are obtained by "separation of variables."

Proof of Theorem 7.9. By Lemma 7.7 we have

$$\psi_m u_{m,\kappa,\alpha} = 0$$

for $\kappa \in \mathcal{K}$. We now show that if $u_m \in \mathcal{U}_\rightarrow$ or $u_m \in \mathcal{U}_\leftarrow$, then $u_m \in \mathcal{U}_\sim \cdot \mathcal{U}_{\pm 1}$, respectively, which will end the proof of assertion 1). Indeed, let us just treat the case where $u_m \in \mathcal{U}_\rightarrow$. The case $u_m \in \mathcal{U}_\leftarrow$ follows by a similar argument.

Note that

$$0 = \delta_q\psi_m u_m = \psi_a^\mu \psi_q u_m\,,$$

and so by Theorem 6.10, there exist $\alpha_1,\ldots,\alpha_s \in R$ and $\lambda \in R$ such that

$$q\frac{du_m/dq}{u_m} = \psi_q u_m = \lambda + \sum_{i=1}^{s} u^\mu_{a,\kappa_i,\alpha_i} = \lambda + \sum_{i=1}^{s} q\frac{du_{m,\kappa_i,\alpha_i}/dq}{u_{m,\kappa_i,\alpha_i}}\,,$$

where $\mathcal{K}_+ = \{\kappa_1,\ldots,\kappa_s\}$. Hence, if

$$v := \frac{u_m}{u_{m,\kappa_1,\alpha_1}\ldots u_{m,\kappa_s,\alpha_s}} = \sum_{n=-\infty}^{\infty} b_n q^n\,,$$

then

$$q\frac{dv/dq}{v} = \lambda\,,$$

and so $nb_n = \lambda b_n$ for all n. Thus, $v = bq^n$ with $n \in \mathbb{Z}$ and $b \in R^\times$. This shows that $u_m \in \mathcal{U}_\sim \cdot \mathcal{U}_1$, which completes the argument.

We now prove assertion 2) for the case where $u_m \in \mathcal{U}_1$. The case $u_m \in \mathcal{U}_{-1}$ is treated by a similar argument. We write

$$u_m = bq^n \prod u_{m,\kappa_i,\alpha_i}$$

as above. Since $u_m(0) = 1 = \prod u_{m,\kappa_i,\alpha_i}(0)$, it follows that $n = 0$ and $b = 1$ so $u_m = \prod u_{m,\kappa_i,\alpha_i}$. This representation is unique due to formula (82).

The last assertion of the theorem is clear. □

Corollary 7.11. *Under the hypotheses of Theorem* 7.9*, let* $u \in \mathcal{U}_\pm$*. Then the following hold:*

(1) *The series* $\overline{\psi_q u} \in k[[q^{\pm 1}]]$ *is integral over* $k[q^{\pm 1}]$ *and the field extension* $k(q) \subset k(q, \overline{\psi_q u})$ *is Abelian with Galois group killed by* p*.*
(2) *If the characteristic polynomial of* ψ_m *is unmixed and* $\mathcal{K}_\pm$ *is short then* u *is transcendental over* $K(q)$*.*

Proof. Assume $u \in \mathcal{U}_+$; the case $u \in \mathcal{U}_-$ is similar. By Theorem 7.9 we may write

$$u = a \prod_{\kappa \in \mathcal{K}_+} u_{m,\kappa,\alpha_\kappa} \, ,$$

with $\alpha_\kappa \in R$, $a \in R^\times$. So by Lemma 7.7,

$$\psi_q u = \sum_{\kappa \in \mathcal{K}_+} u^\mu_{a,\kappa,\alpha_\kappa} \, .$$

By Lemma 6.6, assertion 1 follows. To check assertion 2 note that if u were algebraic over $K(q)$ then the same would hold for $\psi_q u = \delta_q u / u$ and we would get a contradiction by Lemma 6.6. □

Corollary 7.12. *Under the hypotheses of Theorem* 7.9*, the maps* $B^0_\pm : \mathcal{U}_{\pm 1} \to R^{\rho_\pm}$ *are* R*-module isomorphisms. Furthermore, for any* $u \in \mathcal{U}_{\pm 1}$ *we have*

$$u = \sum_{\kappa \in \mathcal{K}_\pm} (B^0_\kappa u) \star u_{m,\kappa,1} \, .$$

So, in particular, the "boundary value problem at $q^{\pm 1} = 0$" is well posed. We now address the "boundary value problem at $q \neq 0$", and the "limit at $q = 0$" issue:

Corollary 7.13. *Under the hypotheses of Theorem* 7.9*, the following hold:*

(1) *If* $\mathcal{K}_+ = \{\kappa\}$*, then for any* $q_0 \in p^\nu R^\times$ *with* $\nu \geq 1$*, and any* $g \in 1 + p^{\kappa\nu} R$*, there exists a unique* $u \in \mathcal{U}_1$ *such that* $u(q_0) = g$*.*
(2) *If* $\mathcal{K}_+ = \{1\}$*, then for any* $q_0 \in pR^\times$*, and any* $g \in R^\times$*, there exists a unique* $u \in \mathcal{U}_{tors} \cdot \mathcal{U}_+$ *such that* $u(q_0) = g$*. Furthermore,* $u(0)$ *is the unique root of unity in* R *that is congruent to* $u(q_0)$ *mod* p*.*

Proof. We recall that $\log : 1 + p^N R \to p^N R$ is a bijection for all $N \geq 1$. So, in order to prove assertion 1), we need to check that there is a unique $\alpha \in R$ such that

$$\log u_{m,\kappa,\alpha}(q_0) = \log \; g \, .$$

By the proof of Proposition 7.9, we have that

$$\log \; u_{m,\kappa,\alpha}(q_0) = \sum_{n \geq 0} c_n \alpha^{\phi^n} \, ,$$

where the p-adic valuation of c_n is $\kappa\nu p^n - n$. Hence, by Lemma 6.15, the mapping

$$\begin{array}{ccc} R & \to & p^{\kappa\nu} R \\ \alpha & \mapsto & \sum c_n \alpha^{\phi^n} \end{array}$$

is bijective, and we are done.

In order to prove assertion 2) let $g=\gamma_0\cdot v_0$ with γ_0 be a root of unity, and $v_0\in 1+pR$. By assertion 1), there exists $v\in\mathcal{U}_1$ such that $v(q_0)=v_0$. Hence, if $u:=\gamma_0\cdot v$, then $u(q_0)=g$, which proves the existence part. Now, if $\gamma_1\in\mathcal{U}_{tors}$, $v_1\in\mathcal{U}_1$, and $\gamma_1 v_1(q_0)=\gamma_0 v_0$, we get that $\gamma_1\equiv\gamma_0 \mod p$, so $\gamma_1=\gamma_0$ and $v_1(q_0)=v_0$. By the uniqueness in the first part above, we see that $v_1=v$, which proves the uniqueness part of the assertion.

The claim about $u(0)$ is clear. □

The following Corollary is concerned with the inhomogeneous equation $\psi_m u=\varphi$.

Corollary 7.14. *Let ψ_m be a non-degenerate $\{\delta_p,\delta_q\}$-character of $\mathbb{G}_m$, and let $\varphi\in q^{\pm 1}R[[q^{\pm 1}]]$ be a series whose support is contained in the set $\mathcal{K}'$ of totally non-characteristic integers of ψ_m. Then the following hold:*

1. *The equation $\psi_m u=\varphi$ has a unique solution $u\in\mathbb{G}_m(q^{\pm 1}R[[q^{\pm 1}]])$ such that the support of $\psi_q u$ is disjoint from the set $\mathcal{K}$ of characteristic integers.*
2. *If $\bar{\varphi}\in k[q^{\pm 1}]$, the series $\overline{\psi_q u}\in k[[q^{\pm 1}]]$ is integral over $k[q^{\pm 1}]$, and the field extension $k(q)\subset k(q,\overline{\psi_q u})$ is Abelian with Galois group killed by p.*
3. *If the characteristic polynomial of ψ_m is unmixed and the support of φ is short then u is transcendental over $K(q)$.*

Proof. Let us assume that $\varphi\in qR[[q]]$. The case $\varphi\in q^{\pm 1}R[[q^{\pm 1}]]$ is similar. We express φ as

$$\varphi=\sum_{\kappa\in S}a_\kappa q^\kappa\,,$$

where S is the support of φ and set $\alpha_\kappa:=\kappa(\mu(0,\kappa))^{-1}a_\kappa$. We define

$$u:=\prod_{\kappa\in S}u_{m,\kappa,\alpha_\kappa}\in\mathbb{G}_m(qR[[q]])\,,$$

which converges q-adically. By Lemma 7.7, $\psi_m u=\varphi$. And observe that u is the unique solution in $\mathbb{G}_m(qR[[q]])$ subject to the condition that $\psi_q u$ has support disjoint from $\mathcal{K}$; cf. Lemma 7.12. By Lemma 7.7,

$$\psi_q u=\sum_{\kappa\in S}u_{a,\kappa,\alpha_\kappa}\,,$$

so we may reach the desired conclusion by Lemma 6.6. □

Remark 7.15. The second part of Corollary 7.13 implies that, if ψ_m is a non-degenerate $\{\delta_p,\delta_q\}$-character of $\mathbb{G}_m$, and $\mathcal{K}_+=\{1\}$, then for any $q_0\in pR^\times$ the group homomorphism

$$\begin{array}{rcl} S_{q_0}:\mu(R)\times R & \rightarrow & \mathbb{G}_m(R)=R^\times\\ (\xi,\alpha) & \mapsto & \xi\cdot u_{m,1,\alpha}(q_0)\end{array}$$

is an isomorphism. Thus, for any $q_1,q_2\in pR^\times$, we have an isomorphism

$$S_{q_1,q_2}:=S_{q_2}\circ S_{q_1}^{-1}:\mathbb{G}_m(R)\rightarrow\mathbb{G}_m(R)\,,$$

which should be viewed as the "propagator" associated to ψ_m. Note that if $\zeta\in\mu(R)$ and $q_0\in pR^\times$, by (83),

$$S_{\zeta q_0}(\xi,\alpha)=\xi\cdot u_{m,1,\alpha}(\zeta q_0)=\xi\cdot u_{m,1,\zeta\alpha}(q_0)=S_{q_0}(\xi,\zeta\alpha)\,,$$

and so

$$S_{\zeta q_0}=S_{q_0}\circ M_\zeta\,,$$

where

$$\begin{array}{rcl} M_\zeta : \mu(R) \times R & \rightarrow & \mu(R) \times R \\ M_\zeta(\xi, \alpha) & = & (\xi, \zeta\alpha) \end{array} .$$

Hence, as for the case of $\mathbb{G}_a$, if $\zeta_1, \zeta_2 \in \mu(R)$, we get that

$$S_{\zeta_1 q_0, \zeta_2 q_0} = S_{q_0} \circ M_{\zeta_2/\zeta_1} \circ S_{q_0}^{-1} .$$

In particular,

$$S_{q_0, \zeta_1\zeta_2 q_0} = S_{q_0, \zeta_2 q_0} \circ S_{q_0, \zeta_1 q_0} ,$$

which can be interpreted as a (weak) "Huygens principle."

8. Elliptic curves

We begin this section by proving some general results about the space of $\{\delta_p, \delta_q\}$-characters of an elliptic curve over a general $\{\delta_p, \delta_q\}$-ring A. We determine bases for these spaces; the answer depends on whether certain Kodaira-Spencer classes (an arithmetic one, and a geometric one) vanish or not. We then prove our main results about $\{\delta_p, \delta_q\}$-characters and their solution spaces for Tate curves over $R((q))$, and for elliptic curves over R.

8.1. **General results.** We have as before a fixed $\{\delta_p, \delta_q\}$-ring ring A that is p-adically complete Noetherian integral domain of characteristic zero. For technical reasons, we assume hereafter that the prime p is greater than 3. We consider an elliptic curve E over A, equipped with an invertible 1-form ω. All the definitions and constructions below apply to the pair (E, ω).

Since $p \geq 5$, the integer 6 is invertible in A, and therefore we have unique elements $a_4, a_6 \in A$ such that $4a_4^3 + 27a_6^2 \in A^\times$, and such that, up to isomorphism, the pair (E, ω) consists of the closure in the projective plane over A of the affine plane curve

$$y^2 = x^3 + a_4 x + a_6 , \tag{84}$$

endowed with the 1-form

$$\omega = \frac{dx}{y} . \tag{85}$$

We set

$$N^r := \ker(J^r_{pq}(E) \stackrel{\pi_r}{\rightarrow} \hat{E}). \tag{86}$$

Exactly as in the theory of algebraic group extensions in the last Chapter of [27], we have an exact sequence

$$0 = \mathrm{Hom}(\hat{E}, \hat{\mathbb{G}}_a) \stackrel{\pi_r^*}{\rightarrow} \mathrm{Hom}(J^r_{pq}(E), \hat{\mathbb{G}}_a) \stackrel{\rho}{\rightarrow} \mathrm{Hom}(N^r, \hat{\mathbb{G}}_a) \stackrel{\partial}{\rightarrow} H^1(E, \mathcal{O}) , \tag{87}$$

where ρ is the restriction. Let us recall that $\mathrm{Hom}(J^r_{pq}(E), \hat{\mathbb{G}}_a)$ identifies with the module $\mathbf{X}^r_{pq}(E)$ of $\{\delta_p, \delta_q\}$-characters. We review and use this sequence by closely following [9], pp. 191-196, where the case of $J^r_p(E)$ was considered.

First of all, note that the projection $\pi_r : J^r_{pq}(E) \rightarrow \hat{E}$ has a natural structure of a principal homogeneous space under $\hat{E} \times N^r \rightarrow \hat{E}$. By Proposition 3.6, π_r has local sections in the Zariski topology. We consider a Zariski open covering and a corresponding family of sections of π_r;

$$E = \bigcup U_\mu, \quad s_\mu : \hat{U}_\mu \rightarrow \pi_r^{-1}(\hat{U}_\mu) . \tag{88}$$

We may and assume that there exists an index μ_0 such that the zero section $0 \in E(A)$ belongs to $U_{\mu_0}(A)$, and $s_{\mu_0}(0)=0$. Then the morphism

$$\tau_\mu : \hat{U}_\mu \times N^r \to \pi_r^{-1}(\hat{U}_\mu)\,, \tag{89}$$

which at the level of S-points (S any A-algebra) is given by $(A,B) \mapsto s_\mu(A)+B$, is an isomorphism of principal homogeneous spaces under $\hat{U}_\mu \times N^r \to \hat{U}_\mu$. The isomorphism τ_{μ_0} induces the identity $0 \times N^r \to \pi_r^{-1}(0) = N^r$, and if $T \in \mathcal{O}(U_{\mu_0})$ is an étale coordinate on U_{μ_0}, then we have an induced $\mathcal{O}(\hat{U}_{\mu_0})$-isomorphism

$$\begin{aligned} \tau_{\mu_0}^* : \mathcal{O}(\hat{U}_{\mu_0})[T^{(i,j)}\mid_{1\leq i+j\leq r}]\widehat{\ } &\stackrel{can}{\simeq} \mathcal{O}^r(\pi_r^{-1}(\hat{U}_{\mu_0})) \stackrel{\circ\tau_{\mu_0}}{\to} \\ &\stackrel{\circ\tau_{\mu_0}}{\to} \mathcal{O}^r(\hat{U}_{\mu_0}\times N^r) = \mathcal{O}^r(\hat{U}_{\mu_0})[T^{(i,j)}\mid_{1\leq i+j\leq r}]\widehat{\ }\,, \end{aligned} \tag{90}$$

where *can* is the unique isomorphism sending $T^{(i,j)}$ into $\delta_p^i\delta_q^j T$. Furthermore, $\tau_{\mu_0}^*$ is the identity modulo T.

Let $\hat{U}_{\mu\nu} = \hat{U}_\mu \cap \hat{U}_\nu$. The sections (88) induce maps $s_\mu - s_\nu : \hat{U}_{\mu\nu} \to J^r_{pq}(E)$ that clearly factor through maps

$$s_{\mu\nu} : \hat{U}_{\mu\nu} \to N^r. \tag{91}$$

In particular, at the level of S-points we have $(\tau_\nu^{-1}\circ\tau_\mu)(A,B) = (A, s_{\mu\nu}(A)+B)$.

We define the map ∂ in (87) by attaching to any homomorphism $\Theta : N^r \to \hat{\mathbb{G}}_a$ the cohomology class $\varphi := \partial(\Theta) \in H^1(\hat{E},\mathcal{O}) = H^1(E,\mathcal{O})$ that is represented by the cocycle $\varphi_{\mu\nu} := \Theta \circ s_{\mu\nu} \in \mathcal{O}(\hat{U}_{\mu\nu})$. Then, as in [9], p. 192, we check that the sequence (87) is exact.

Proposition 8.1. *The rank of the A-module $\mathbf{X}^r_{pq}(E)$ is $r(r+3)/2$ if $\partial = 0$, and $r(r+3)/2-1$ if $\partial \neq 0$.*

Proof. Looking at the Lie algebras, it is clear that $\mathrm{Hom}(N^r,\hat{\mathbb{G}}_a)$ is an A-module of rank at most $\dim N^r = r(r+3)/2$. By Lemma 4.6, the homomorphisms $L^{[a,b]} \in \mathrm{Hom}(N^r,\hat{\mathbb{G}}_a)$, $1 \leq a+b \leq r$, are A-linearly independent. Thus, $\mathrm{Hom}(N^r,\hat{\mathbb{G}}_a)$ has rank $r(r+3)/2$ over A, and the conclusion follows. $\square$

Since $H^1(E,\mathcal{O})$ has rank 1 over A, by the exact sequence (87) we conclude that

Corollary 8.2. *The A-module $\mathbf{X}^1_{pq}(E)$ is non-zero.*

This result contrasts deeply with the "ode" story for both, the geometric [4] and arithmetic [5] case. Indeed, for "general" E we then have that

$$\mathbf{X}^1_p(E) = \mathbf{X}^1_q(E) = 0\,.$$

In the sequel, we construct and analyze $\{\delta_p,\delta_q\}$-characters of E in more detail. We use the invertible 1-form ω on E, and assume that the closed subscheme of U_{μ_0} defined by T is the zero section 0. We identify the T-adic completion of $\mathcal{O}(U_{\mu_0})$ with the ring of power series $A[[T]]$, and furthermore, choose T such that $\omega \equiv dT$ mod (T). Using the cubic (84), we may take, for instance, $T = -\frac{x}{y}$. We set $W := -\frac{1}{y}$.

This affine coordinate T, and W around the zero section, are mapped into $T \in A[[T]]$ and

$$W(T) = T^3 + a_4T^7 + \cdots \in A[[T]]\,, \tag{92}$$

respectively; cf. [29], p. 111.

For all integers $a,b \in \mathbb{Z}_+$ with $a+b \geq 1$, we define $\{\delta_p,\delta_q\}$-*Kodaira-Spencer classes* $f^{[a,b]} \in A$ as follows. For $r \geq a+b$, we have a natural isomorphism $N^r \simeq$

$(\hat{\mathbb{A}}^{\frac{r(r+3)}{2}},[+])$, where $\hat{\mathbb{A}}^{\frac{r(r+3)}{2}}=Spf\ A[T^{(i,j)}\mid_{1\leq i+j\leq r}]\widehat{\ }$, which we view hereafter as an identification. Thus, we obtain the identification

$$s_{\mu\nu}=(\alpha_{\mu\nu}^{i,j}\mid_{1\leq i+j\leq r})\in\mathcal{O}(\hat{U}_{\mu\nu})^{\frac{r(r+3)}{2}}\,,\\ \alpha_{\mu\nu}^{i,j}:=T^{(i,j)}\circ s_{\mu\nu}\,.$$

By (41), we have the series $L^{[a,b]}\in A[T^{(i,j)}\mid_{1\leq i+j\leq r}]\widehat{\ }$ defining homomorphisms

$$L^{[a,b]}:N^r\simeq(\hat{\mathbb{A}}^{\frac{r(r+3)}{2}},[+])\to\hat{\mathbb{G}}_a=(\hat{\mathbb{A}}^1,+)\,.$$

We define elements

$$\varphi_{\mu\nu}^{[a,b]}:=L^{[a,b]}(\alpha_{\mu\nu}^{i,j}\mid_{1\leq i+j\leq r})\in\mathcal{O}(\hat{U}_{\mu\nu})\,.$$

As we vary $\mu\nu$, the collection of such sections defines a cohomology class

$$\varphi^{[a,b]}\in H^1(\hat{E},\mathcal{O}_{\hat{E}})=H^1(E,\mathcal{O}_E)\,,$$

which is, of course, the class $\gamma(L^{[a,b]})$ defined by the exact sequence (87). Let

$$\langle\ ,\ \rangle:H^1(E,\mathcal{O})\times H^0(E,\Omega^1)\to A$$

denote the Serre duality pairing, and define

$$f^{[a,b]}:=\langle\varphi^{[a,b]},\omega\rangle\in A\,. \tag{93}$$

It is clear that $f^{[a,b]}$ depends only on a and b but not on r. It is also clear that ∂ in the exact sequence (87) is 0 if and only if $f^{[a,b]}=0$ for all $1\leq a+b\leq r$. Proceeding verbatim as in [9], we check that $f^{[a,b]}$ depends on the pair (E,ω), and not on the choice of T, as long as T satisfies the condition that $\omega\equiv dT$ mod (T).

Now let $a,b,c,d\in\mathbb{Z}_+$ with $1\leq a+b,c+d\leq r$, and and consider the homomorphism

$$\Theta:=\Theta_{[c,d]}^{[a,b]}:N^r\to\hat{\mathbb{G}}_a$$

given by

$$\Theta:=f^{[a,b]}L^{[c,d]}-f^{[c,d]}L^{[a,b]}\in A[T^{(i,j)}\mid_{1\leq i,j\leq r}]\widehat{\ }\,.$$

Then we have that $\partial(\Theta)=0$, for $\partial(\Theta)$ is the class $[\gamma_{\mu\nu}]$ in $H^1(\hat{E},\mathcal{O})$ of the cocycle

$$\gamma_{\mu\nu}:=f^{[a,b]}\varphi_{\mu\nu}^{[c,d]}-f^{[c,d]}\varphi_{\mu\nu}^{[a,b]}\in\mathcal{O}(\hat{U}_{\mu\nu})\,,$$

and $[\gamma_{\mu\nu}]=0$ because

$$\langle[\gamma_{\mu\nu}],\omega\rangle=f^{[a,b]}\cdot\langle\varphi^{[c,d]},\omega\rangle-f^{[c,d]}\cdot\langle\varphi^{[a,b]},\omega\rangle=0\,.$$

By the exactness of the sequence (87), Θ lifts to a unique homomorphism

$$\psi=\psi_{[c,d]}^{[a,b]}:J^r_{pq}(E)\to\hat{\mathbb{G}}_a$$

which we interpret as a $\{\delta_p,\delta_q\}$-character

$$\psi\in\mathbf{X}^r_{pq}(E)\,.$$

This character ψ depends only on a,b,c,d, but not on r in the sense that if we change r to $r+s$, then the new ψ is obtained from the old one by composition with the projection $J^{r+s}_{pq}(E)\to J^r_{pq}(E)$. Incidentally, ψ is obtained by gluing functions

$$\psi_\mu\circ\tau_\mu^{-1}\in\mathcal{O}(\pi_r^{-1}(\hat{U}_\mu))\,, \tag{94}$$

$$\psi_\mu:=f^{[a,b]}\cdot L^{[c,d]}-f^{[c,d]}\cdot L^{[a,b]}+\gamma_\mu\in\mathcal{O}(\hat{U}_\mu)[T^{(i,j)}\mid_{1\leq i+j\leq r}]\widehat{\ }\,, \tag{95}$$

with $\gamma_\mu \in \mathcal{O}(\hat{U}_\mu)$, and we have the identities

$$\begin{array}{rcl} \psi^{[a,b]}_{[a,b]} & = & 0\,, \\ \psi^{[a,b]}_{[c,d]} + \psi^{[c,d]}_{[a,b]} & = & 0\,, \\ f^{[a_1,b_1]}\psi^{[a_2,b_2]}_{[a_3,b_3]} + f^{[a_2,b_2]}\psi^{[a_3,b_3]}_{[a_1,b_3]} + f^{[a_3,b_3]}\psi^{[a_1,b_1]}_{[a_2,b_2]} & = & 0\,, \end{array}$$

which follow from the very same identities that are obtained when the ψs are replaced by the Θs.

Since τ_{μ_0} is the identity modulo T, we have that $\psi \circ e(pT)$ is congruent to $\psi_{\mu_0} \circ e(pT)$ modulo T. By Lemmas 4.5 and 4.2, we obtain

$$\left(\psi^{[a,b]}_{[c,d]}\right) \circ e(pT) = p^{1+\epsilon(d)} f^{[a,b]} \phi_p^c \delta_q^d T - p^{1+\epsilon(b)} f^{[c,d]} \phi_p^a \delta_q^b T + \tilde{f}^{[a,b]}_{[c,d]} T\,, \tag{96}$$

where $\tilde{f}^{[a,b]}_{[c,d]} \in A$. So $\psi^{[a,b]}_{[c,d]}$, viewed as an element of $A[[T]][\delta_p^i \delta_q^j T \mid_{1\leq i+j\leq r}]\widehat{\ }$, has the form

$$\psi^{[a,b]}_{[c,d]} = \frac{1}{p}\left[p^{1+\epsilon(d)} f^{[a,b]} \phi_p^c \delta_q^d - p^{1+\epsilon(b)} f^{[c,d]} \phi_p^a \delta_q^b + \tilde{f}^{[a,b]}_{[c,d]}\right] l(T)\,. \tag{97}$$

Remark 8.3. We will use the following notation:

$$\begin{array}{lclllclllcl} f_p^a & := & f^{[a,0]}\,, & f_q^a & := & f^{[0,a]}\,, \\ \tilde{f}_p^2 & := & \tilde{f}^{[1,0]}_{[2,0]}\,, & \tilde{f}_q^2 & := & \tilde{f}^{[0,1]}_{[0,2]}\,, & \tilde{f}_{pq}^1 & := & \tilde{f}^{[1,0]}_{[0,1]}\,. \end{array}$$

The elements $f_p^a \in A$ are (the images of) the elements f^a_{jet} in [9]. The elements $\tilde{f}_p^2$ are (the images of) the elements $pf^{1,2}_{jet}$ in [9]. By [9], Proposition 7.20 (and Remark 7.21) and Corollary 8.84 (and Remark 8.85), we have

$$\tilde{f}_p^2 = p(f_p^1)^\phi\,. \tag{98}$$

The element f_p^1 was interpreted in [9] as an *arithmetic Kodaira-Spencer class* of E. The element f_q^1 is easily seen to be (an incarnation of) the usual Kodaira-Spencer class of E; cf. [7]. The element $f_q^1 \in A$, and the reduction mod p of f_p^1, were explicitly computed in [18]. The δ_q-character

$$\psi_q^2 := \psi^{[0,1]}_{[0,2]} \in \mathbf{X}_q^2(E)$$

is (an "incarnation" of) the *Manin map* of E [22], constructed as in [2]. If $f_q^1 \neq 0$, then $\psi_q^2 \neq 0$. (Unlike the construction in [2] that was done over a field, our construction here is carried over the ring A.) The δ_p-character

$$\psi_p^2 := \psi^{[1,0]}_{[2,0]} \in \mathbf{X}_p^2(E)$$

is the *arithmetic Manin map* in [5]. And if $f_p^1 \neq 0$, then $\psi_p^2 \neq 0$. Both of these Manin maps are "ode" maps with respect to the geometric and the arithmetic direction separately. On the other hand, we can consider the $\{\delta_p, \delta_q\}$-character

$$\psi_{pq}^1 := \psi^{[1,0]}_{[0,1]} \in \mathbf{X}^1_{pq}(E)\,.$$

If either $f_p^1 \neq 0$ or $f_q^1 \neq 0$, then $\psi_{pq}^1 \neq 0$. If both $f_p^1 \neq 0$ and $f_q^1 \neq 0$, then ψ^1_{pq} is a "purely pde" operator (in the sense that it is not a sum of a δ_p-character and a δ_q-character).

Indeed, we have the following consequence of Proposition 8.1.

Corollary 8.4. *If $f_p^1 \neq 0$ and $f_q^1 \neq 0$, then:*

(1) ψ^1_{pq} *is an* L*-basis of* $\mathbf{X}^1_{pq}(E)\otimes L$.
(2) $\mathbf{X}^1_p(E)=\mathbf{X}^1_q(E)=0$.

We may view ψ^1_{pq} as a canonical "convection equation" on E.

Using the notation above and the commutation relations in $A[\phi_p,\delta_q]$, we have the following equalities:

(99)

$$\begin{array}{rcl}
\psi^1_{pq} & = & \frac{1}{p}\left[pf^1_p\delta_q - f^1_q\phi_p + \tilde{f}^1_{pq}\right] l(T)\,,\\
\delta_q\psi^1_{pq} & = & \frac{1}{p}\left[p(\delta_q f^1_p)\delta_q + pf^1_p\delta_q^2 - (\delta_q f^1_q)\phi_p - f^1_q p\phi_p\delta_q + \delta_q\tilde{f}^1_{pq} + \tilde{f}^1_{pq}\delta_q\right] l(T)\,,\\
\phi_p\psi^1_{pq} & = & \frac{1}{p}\left[p(f^1_p)^\phi\phi_p\delta_q - (f^1_q)^\phi\phi_p^2 + (\tilde{f}^1_{pq})^\phi\phi_p\right] l(T)\,,\\
\psi^2_p & = & \frac{1}{p}\left[f^1_p\phi_p^2 - f^2_p\phi_p + \tilde{f}^2_p\right] l(T)\,,\\
\psi^2_q & = & \frac{1}{p}\left[pf^1_q\delta_q^2 - pf^2_q\delta_q + \tilde{f}^2_q\right] l(T)\,.
\end{array}$$

Thus, if we represent the ordered elements $\delta_q\psi^1_{pq},\ \phi_p\psi^1_{pq},\ \psi^2_p,\ \psi^1_{pq},\ \psi^2_q$ as L-linear combinations of the series $p\delta_q^2 l(T),\ p\phi_p\delta_q l(T),\ \phi_p^2 l(T),\ p\delta_q l(T),\ \phi_p l(T),\ l(T)$, then the matrix of L-coefficients is equal to $\frac{1}{p}M$, where

$$M := \left(\begin{array}{cccccc}
f^1_p & -f^1_q & 0 & \delta_q f^1_p + \frac{\tilde{f}^1_{pq}}{p} & -\delta_q f^1_q & \delta_q\tilde{f}^1_{pq}\\
0 & (f^1_p)^\phi & -(f^1_q)^\phi & 0 & (\tilde{f}^1_{pq})^\phi & 0\\
0 & 0 & f^1_p & 0 & -f^2_p & \tilde{f}^2_p\\
0 & 0 & 0 & f^1_p & -f^1_q & \tilde{f}^1_{pq}\\
f^1_q & 0 & 0 & -f^2_q & 0 & \tilde{f}^2_q
\end{array}\right).$$

Proposition 8.5. *Let us assume that* $f^1_p\neq 0$ *and* $f^1_q\neq 0$*. Then the following hold:*

1) *The elements*
$$\psi^1_{pq},\ \delta_q\psi^1_{pq},\ \phi_p\psi^1_{pq},\ \psi^2_p$$
form an L*-basis of* $\mathbf{X}^2_{pq}(E)\otimes L$.
2) *The elements*
$$\psi^1_{pq},\ \phi_p\psi^1_{pq},\ \psi^2_q,\ \psi^2_p$$
form an L*-basis of* $\mathbf{X}^2_{pq}(E)\otimes L$.
3) *There exists a* 5*-tuple* $(\alpha,\beta,\gamma,\nu,\lambda)\in A^5$*, which is unique up to scaling by an element of* A*, satisfying*
$$(\alpha\delta_q+\beta\phi_p+\gamma)\psi^1_{pq} = \nu\psi^2_q + \lambda\psi^2_p,\quad \alpha\neq 0,\ \nu\neq 0\,.$$
4) *All* 5×5 *minors of the matrix* M *vanish.*

We may view the character $\nu\psi^2_q+\lambda\psi^2_p$ as a canonical "wave equation" on E.

Proof. By the form of the matrix M, each of the 4-tuples in assertions 1) and 2) are L-linearly independent. By Proposition 8.1, $\mathbf{X}^2_{pq}(E)$ has rank 4 over A. All the statements in the Proposition then follow. □

Similar arguments (cf. also to the case of $\mathbb{G}_m$) yield:

Proposition 8.6. *Let us assume that $f_p^1 \neq 0$ and $f_q^1 \neq 0$. Then, for any $r \geq 2$, the L-vector space $\mathbf{X}^r_{pq}(E) \otimes L$ has basis*

$$\{\phi_p^i \delta_q^j \psi^1_{pq} \mid_{0 \leq i+j \leq r-1}\} \cup \{\phi_p^i \psi_p^2 \mid_{0\leq i \leq r-2}\}\,.$$

When either $f_p^1 = 0$ or $f_q^1 = 0$, the picture above changes, and in fact, it simplifies. For if with more generality we assume that $f^{[a,b]} = 0$ for some a, b with $a+b \leq r$, then $0 = \varphi^{[a,b]} = \gamma(L^{[a,b]})$, and by the exact sequence (87), $L^{[a,b]}$ lifts uniquely to a homomorphism $\psi^{[a,b]} : J^r_{pq}(E) \rightarrow \hat{\mathbb{G}}_a$, which we interpret as a $\{\delta_p, \delta_q\}$-character $\psi^{[a,b]} \in \mathbf{X}^r_{pq}(E)$. Let us observe in passing that $\psi^{[a,b]}$ is obtained by gluing functions

$$\begin{array}{c} \psi_\mu \circ \tau_\mu^{-1} \in \mathcal{O}(\pi_r^{-1}(\hat{U}_\mu))\,, \\ \psi_\mu := L^{[a,b]} + \gamma_\mu \in \mathcal{O}(\hat{U}_\mu)[T^{(i,j)} \mid_{1 \leq i+j \leq r}]\,\widehat{}\,, \end{array}$$

with $\gamma_\mu \in \mathcal{O}(\hat{U}_\mu)$. As before, we obtain

$$\psi^{[a,b]} \circ e(pT) = p^{1+\epsilon(b)} \phi_p^a \delta_q^b T + \tilde{f}^{[a,b]}$$

where $\tilde{f}^{[a,b]} \in A$. So $\psi^{[a,b]}$, viewed as a series, has the form

$$\psi^{[a,b]} = \frac{1}{p}\left[p^{1+\epsilon(b)} \phi_p^a \delta_q^b + \tilde{f}^{[a,b]}\right] l(T)\,.$$

Incidentally, if $f_q^1 = 0$, then

$$\psi_q^1 := \psi^{[0,1]} \in \mathbf{X}^1_q(E) \tag{100}$$

is (an "incarnation" of) the *Kolchin logarithmic derivative* of E [20]; cf. Proposition 8.29 below.

On the other hand, if $f_p^1 = 0$ we set

$$\psi_p^1 := \psi^{[1,0]} \in \mathbf{X}_p(E)\,. \tag{101}$$

As before, we find the following bases for the set of $\{\delta_p, \delta_q\}$-characters:

Proposition 8.7. *Let us assume that $f_p^1 = 0$ and $f_q^1 \neq 0$. Then, for each $r \geq 2$, the L-vector space $\mathbf{X}^r_{pq}(E) \otimes L$ has basis*

$$\{\phi_p^i \delta_q^j \psi_q^2 \mid {}_{0 \leq i+j \leq r-2}\} \cup \{\phi_p^i \psi_p^1 \mid_{0 \leq i \leq r-1}\} \cup \{\phi_p^i \delta_q \psi_p^1 \mid_{0 \leq i \leq r-2}\}\,.$$

Proposition 8.8. *Let us assume that $f_p^1 \neq 0$ and $f_q^1 = 0$. Then, for each $r \geq 2$, the L-vector space $\mathbf{X}^r_{pq}(E) \otimes L$ has basis*

$$\{\phi_p^i \psi_p^2 \mid_{0 \leq i \leq r-2}\} \cup \{\phi_p^i \delta_q^j \psi_q^1 \mid_{0 \leq i+j \leq r-1}\}\,.$$

Proposition 8.9. *Let us assume $f_p^1 = 0$ and $f_q^1 = 0$. Then, for each $r \geq 1$, the L-vector space $\mathbf{X}^r_{pq}(E) \otimes L$ has basis*

$$\{\phi_p^i \psi_p^1 \mid_{0 \leq i \leq r-1}\} \cup \{\phi_p^i \delta_q^j \psi_q^1 \mid_{0 \leq i+j \leq r-1}\}\,.$$

8.2. **Tate curves.** Let E_q be the *Tate curve with parameter* q over $A := R((q))\widehat{\ }$, equipped with its canonical 1-form ω_q. This curve E_q is defined as the elliptic curve in the projective plane over A whose affine plane equation is

$$y^2 = x^3 - \frac{1}{48}E_4(q)x + \frac{1}{864}E_6(q)\,,$$

where E_4 and E_6 are the Eisenstein series

$$\begin{array}{rcl} E_4(q) & = & 1 + 240 \cdot s_3(q)\,, \\ E_6(q) & = & 1 - 504 \cdot s_5(q)\,. \end{array}$$

In here, for $m \geq 1$, we follow the usual convention and write

$$s_m(q) := \sum_{n\geq 1} \frac{n^m q^n}{1-q^n} \in R[[q]]\,.$$

Also, the canonical form is defined by

$$\omega_q = \frac{dx}{y}\,.$$

More generally, let $\beta \in R^\times$ be an invertible element that we shall view as a varying parameter, and let $(E_{\beta q}, \omega_{\beta q})$ be the pair obtained by base change from (E_q, ω_q) via the isomorphism

$$\begin{array}{rcl} R((q))\widehat{\ } & \stackrel{\sigma_\beta}{\to} & R((q))\widehat{\ } \\ \sigma_\beta(\sum a_n q^n) & = & \sum a_n \beta^n q^n \end{array}\,.$$

Thus, the pair $(E_{\beta q}, \omega_{\beta q})$ is the elliptic curve

$$y^2 = x^3 - \frac{1}{48}E_4(\beta q)x + \frac{1}{864}E_6(\beta q)$$

equipped with the 1-form

$$\omega_{\beta q} = \frac{dx}{y}\,.$$

(Let us observe that σ_β is a δ_q-ring homomorphism, but not a δ_p-ring homomorphism.) We shall refer to $(E_{\beta q}, \omega_{\beta q})$ as the *Tate curve with parameter* βq.

Let us observe that the discussion and notation in Example 2.16 that concerns groups of solutions does not apply to $E_{\beta q}$ over A, as $A = R((q))\widehat{\ } \neq R$.

From now on, we assume that all our quantities in the general theory (like the f's, the $\tilde{f}$'s, and the ψ's) are associated to the pair $(E_{\beta q}, \omega_{\beta q})$.

Lemma 8.10. *There exists* $c \in \mathbb{Z}_p^\times$ *such that, for any* $\beta \in R^\times$*, we have*

$$f^1_p = \eta, \quad f^2_p = \eta^\phi + p\eta, \quad \tilde{f}^2_p = p\eta^\phi\,,$$

where

$$\eta := \frac{c}{p}\log\frac{\phi(\beta)}{\beta^p}\,.$$

Proof. We consider the ring $R((q))\widehat{\ }[q', q'', \ldots]\widehat{\ }$ with its unique δ_p-ring structure such that $\delta_p q = q'$, $\delta_p q' = q''$, etc. Let us note that $(E_{\beta q}, \omega_{\beta q})$ is obtained by base change from (E_q, ω_q) via the composition

$$R((q))\widehat{\ } \subset R((q))\widehat{\ }[q', q'', \ldots]\widehat{\ } \stackrel{\tilde{\sigma}_\beta}{\to} R((q))\widehat{\ }[q', q'', \ldots]\widehat{\ } \stackrel{\pi}{\to} R((q))\widehat{\ }\,, \tag{102}$$

where $\tilde{\sigma}_\beta(q) = \beta q$, $\tilde{\sigma}_\beta(q') = \delta_p(\beta q)$, $\tilde{\sigma}_\beta(q'') = \delta_p^2(\beta q), \ldots$, and $\pi(q) = q$, $\pi(q') = 0$, $\pi(q'') = 0, \ldots$ And let us note also that $\tilde{\sigma}_\beta$ and π are δ_p-ring homomorphisms. By

the functoriality of f_p^r plus [7], Corollary 7.26, [1], Corollary 6.1, and [8], Lemma 6.14, there exists $c \in \mathbb{Z}_p^\times$ such that, for any β, we have

$$\begin{aligned} f_p^1 &= \frac{c}{p} \cdot \log \frac{\phi_p(\beta q)}{(\beta q)^p} = \frac{c}{p} \log \frac{\phi_p(\beta)}{\beta^p}\,, \\ f_p^2 &= (f_p^1)^\phi + p f_p^1\,. \end{aligned}$$

Thus, f_p^1 and f_p^2 have the desired values. The value of $\tilde{f}_p^2$ follows by (98). $\square$

Lemma 8.11. *The following hold:*

1) f_q^1 *does not depend on* β.
2) $f_q^1 \in \mathbb{Z}_p^\times$, *and* $f_q^2 = \tilde{f}_q^2 = 0$.

Proof. We start by noting that $(E_{\beta q}, \omega_{\beta q})$ is obtained by a base change from (E_q, ω_q) via the isomorphism $\sigma_\beta : R((q))\hat{} \to R((q))\hat{}$, and recall that σ_β is a δ_q-ring homomorphism. By functoriality, the elements f_q^1, f_q^2, $\tilde{f}_q^2$ corresponding to a given β are the images via σ_β of the corresponding quantities for $\beta = 1$. Thus, proving assertion 2) for $\beta = 1$ suffices to conclude both assertions 1) and 2) for arbitrary β.

We prove next assertion 2) for $\beta = 1$.

The fact that $f_q^1 \in \mathbb{Z}_p^\times$ follows from [7], Corollary 7.26. In order to prove that $f_q^2 = \tilde{f}_q^2 = 0$, it is enough to show that there exists $\psi \in \mathbf{X}_q^2(E)$ such that $\psi = \delta_q^2 l(T)$. This is because $\mathbf{X}_q^2(E)$ has rank 1 over A [4].

The existence of ψ can be argued analytically, and we briefly sketch the argument next. (A purely algebraic argument is also available but the analytic one is simpler and classical, going back to Fuchs and Manin [22].) For in our problem we can replace $R((q))$ by $\mathbb{Q}((q))$, and then $\mathbb{Q}((q))$ by $\mathbb{C}((q))$. We view the Tate curve over $\mathbb{C}((q))$ as arising from an analytic family $E_q^{an} \to \Delta^*$ of elliptic curves over the punctured disk Δ^*. The fiber E_{q_τ} of this family over a point $q_\tau = e^{2\pi i \tau} \in \Delta^*$ identifies with $\mathbb{C}/\langle 1, \tau\rangle$. Let z be the coordinate on $\mathbb{C}$. Then, for any local analytic section P, $q_\tau \mapsto P(q_\tau)$, of $E_q^{an} \to \Delta^*$ that is close to the zero section $q_\tau \mapsto P_0(q_\tau) = 0$, we have that

$$l(P(q_\tau)) = \int_{P_0(q_\tau)}^{P(q_\tau)} \omega_{q_\tau} \tag{103}$$

where ω_{q_τ} is the 1-form on E_{q_τ} whose pull back to $\mathbb{C}$ is dz. The periods of ω_{q_τ} on E_{q_τ} are 1 and τ, so they are annihilated by

$$\left(\frac{d}{d\tau}\right)^2 = -4\pi^2 \left(q_\tau \frac{d}{dq_\tau}\right)^2 .$$

Hence the map

$$P \mapsto \left(q_\tau \frac{d}{dq_\tau}\right)^2 \left(\int_{P_0(q_\tau)}^{P(q_\tau)} \omega_{q_\tau}\right)$$

is well defined for all local analytic sections P of $E_q^{an} \to \Delta^*$ (not necessarily close to the zero section P_0). This map arises from a δ_q-character ψ of E_q over $\mathbb{C}((q))$ [22]. By (103), ψ coincides with $P \mapsto \delta_q^2 l(P)$ for P close to the zero section, and this completes the argument. $\square$

Remark 8.12. From now on, we choose c as in Lemma 8.10, and we set

$$\begin{array}{rcl} \eta & := & f^1_p = \frac{c}{p}\log \frac{\phi_p(\beta)}{\beta^p} \in R\,, \\ \gamma & := & f^1_q \in \mathbb{Z}_p^\times\,. \end{array}$$

Notice that we have $\eta = 0$ if, and only if, β is a root of unity. Later on, the quotient of these Kodaira-Spencer classes,

$$\frac{\eta}{\gamma} = \frac{f^1_q}{f^1_p}\,,$$

will play a key rôle.

Lemma 8.13. $\tilde{f}^1_{pq} = p\gamma$.

Proof. Let us assume first that β is not a root of unity, and so, $\eta \neq 0$. The 5×5 minor of the matrix M above obtained by removing the 6-th column has determinant zero (cf. Proposition 8.5), and using Lemma 8.10, Lemma 8.11, and Remark 8.12, we get the condition

$$pf^1_p(\tilde{f}^1_{pq} - p\gamma)^\phi + (f^1_p)^\phi(\tilde{f}^1_{pq} - p\gamma) = 0\,.$$

We take the valuation $v_p : R((q))\widehat{\ } \to \mathbb{Z}_+ \cup \{\infty\}$ in this identity, and use the fact that $v_p \circ \phi = v_p$ to get $\tilde{f}^1_{pq} - p\gamma = 0$, completing the proof of this case.

If instead now $\beta \in R^\times$ is arbitrary, proceeding as in the proof of Lemma 8.10, we have that $\tilde{f}^1_{pq}$ is the image of an element $F \in R((q))\widehat{\ }[q', q'', \ldots]\widehat{\ }$ via the map $\pi \circ \tilde{\sigma}_\beta$ in (102). We know that

$$\pi\tilde{\sigma}_\beta F = p\gamma \tag{104}$$

for all $\beta \in R^\times \backslash \mu(R)$. Since the map

$$\begin{array}{rcl} R^\times & \to & R((q))\widehat{\ } \\ \beta & \mapsto & \pi\tilde{\sigma}_\beta F \end{array}$$

is p-adically continuous, we have that that (104) holds for all $\beta \in R^\times$. This completes the proof. $\square$

Combining Lemma 8.10, Lemma 8.11, and (99), we derive the following.

Proposition 8.14. *Let $E = E_{\beta q}$ be the Tate curve over $A = R((q))\widehat{\ }$ with parameter βq, $\beta \in R^\times$. When viewed as elements of the ring*

$$A[[T]][\delta_p T, \delta_q T, \delta_p^2 T, \delta_p\delta_q T, \delta_q^2 T]\widehat{\ }\,,$$

the characters ψ^1_{pq}, ψ^2_q, ψ^2_p are equal to

$$\begin{array}{rcl} \psi^1_{pq} & = & \dfrac{1}{p}[p\eta\delta_q - \gamma\phi_p + p\gamma]l(T)\,, \\ \psi^2_q & = & \gamma\delta_q^2 l(T)\,, \\ \psi^2_p & = & \dfrac{1}{p}[\eta\phi_p^2 - (\eta^\phi + p\eta)\phi_p + p\eta^\phi]l(T)\,. \end{array}$$

In particular, we have the relations

$$\begin{array}{rcl} (\gamma\eta^{\phi+1}\delta_q + \gamma^2\eta\phi_p - \gamma^2\eta^\phi)\psi^1_{pq} & = & \eta^{\phi+2}\psi^2_q - \gamma^3\psi^2_p\,, \\ \gamma\delta_q^2\psi^1_{pq} & = & (\eta\delta_q - p\gamma\phi_p + \gamma)\psi^2_q\,. \end{array}$$

Remark 8.15. Of course, if β is a root of unity, the first of the last two relations in the Proposition above reduces to the identity $0=0$. In this case, we also have that

$$\psi^1_{pq} = -\gamma \psi^1_p \, . \tag{105}$$

Remark 8.16. By the Proposition above, the Fréchet symbol of ψ^1_{pq} with respect to ω is $\theta_{\psi^1_{pq},\omega} = \eta\xi_q - \gamma\xi_p + \gamma$. Thus, if we consider the "simplest" of the energy functions on E given by

$$H := (\psi^1_{pq})^2 \, ,$$

a direct computation shows the following expression for the Euler-Lagrange equation attached to H and the vector field ∂ dual to ω:

$$\epsilon^1_{H,\partial} = (-2\eta^{\phi}\phi_p\delta_q + 2\gamma\phi_p - 2\gamma)\psi^1_{pq} \, .$$

In particular, any solution of ψ^1_{pq} is a solution of the Euler Lagrange equation $\epsilon^1_{H,\partial}$.

In the sequel, we will need to use some other facts about Tate's curves that we now recall. Indeed, note that the cubic defining $E_{\beta q}$ has coefficients in $R[[q]]$, and hence, $E_{\beta q}$ has a natural projective (non-smooth) model $\mathcal{E} = \mathcal{E}_{\beta q}$ over $R[[q]]$ equipped with a "zero" section defined by $T = -\frac{x}{y}$. The completion of $\mathcal{E}_{\beta q}$ along this section is naturally isomorphic to $Spf\ R[[q]][[T]]$, where $R[[q,T]] = R[[q]][[T]]$ is viewed with its T-adic topology. This isomorphism induces a natural embedding

$$\iota : qR[[q]] \rightarrow \mathcal{E}(R[[q]]) \subset E(A) \, ,$$

that is explicitly given by sending any $u \in qR[[q]]$ into the point of $\mathcal{E}$ whose (T,W)-coordinates are

$$(u, u^3 - \frac{1}{48}E_4(\beta q)u^7 + \cdots) \, ,$$

cf. (92). We denote by $E_1(A)$ the image of this embedding, that is to say, $E_1(A) = \iota(qR[[q]])$. When no confusion can arise, we will view ι as an inclusion, and we will identify $\iota(qR[[q]])$ with $qR[[q]]$.

The formal group law $\mathcal{F} = \mathcal{F}_E = \mathcal{F}(T_1,T_2)$ of E with respect to $T = -\frac{x}{y}$ has coefficients in $R[[q]]$, and is isomorphic over $R[[q]]$ to the formal group $\mathcal{F}_{\mathbb{G}_m} = t_1 + t_2 + t_1t_2$ of $\mathbb{G}_m$ via an isomorphism $\sigma(t) = t + \cdots \in t + t^2R[[q,t]])$.

For convenience, we recall how σ arises. We first notice that for any variable v, the series

$$\begin{array}{rcl} X(q,v) & := & \frac{v}{(1-v)^2} + \sum_{n\geq 1}\left(\frac{\beta^n q^n v}{(1-\beta^n q^n v)^2} + \frac{\beta^n q^n v^{-1}}{(1-\beta^n q^n v^{-1})^2} - 2\frac{\beta^n q^n}{(1-\beta^n q^n)^2}\right) \, , \\ Y(q,v) & := & \frac{v^2}{(1-v)^3} + \sum_{n\geq 1}\left(\frac{(\beta^n q^n v)^2}{(1-\beta^n q^n v)^3} - \frac{\beta^n q^n v^{-1}}{(1-\beta^n q^n v^{-1})^3} + \frac{\beta^n q^n}{(1-\beta^n q^n)^2}\right) \, , \end{array} \tag{106}$$

make sense as elements of the ring $\mathbb{Z}[v,v^{-1}][[q]][\frac{1}{1-v}]$. Cf. [29], p. 425. So, if we specialize $v \mapsto 1+t \in \mathbb{Z}[[t]]$ where t is a variable, then X and Y become elements in $\mathbb{Z}[[q]]((t))$, that is to say, Laurent series in t with coefficients in $\mathbb{Z}[[q]]$. Since $p \geq 5$, the series

$$\begin{array}{rcl} x & = & X + \frac{1}{12} \\ y & = & -Y - \frac{X}{2} \end{array}$$

belong to $\mathbb{Z}_{(p)}[[q]]((t))$. Note that $X = t^{-2} + \cdots$ and $Y = t^3 + \cdots$, hence, $x = t^{-2} + \cdots$ $y = -t^3 + \cdots$ and so $T = -\frac{x}{y} = t + \cdots =: \sigma(t) \in t + t^2\mathbb{Z}_{(p)}[[q,t]]$. It turns out that $t \mapsto T = \sigma(t)$ defines an isomorphism $\mathcal{F}_{\mathbb{G}_m} \simeq \mathcal{F}_E$; cf. [29], p. 431.

Definition 8.17. The *characteristic polynomial* $\mu(\xi_p, \xi_q)$ of a character ψ^1_{pq} is the Fréchet symbol of ψ^1_{pq} with respect to ω:

$$\mu(\xi_p, \xi_q) := \eta\xi_q - \gamma\xi_p + \gamma\,.$$

Note that $\mu(0,0) \in R^\times$. The *characteristic integers* of ψ^1_{pq} are the integers κ such that $\mu(0,\kappa) = 0$, that is to say, solutions of $\eta\kappa = -\gamma$. The set $\mathcal{K}$ of characteristic integers has at most one element and is given by

$$\mathcal{K} = \{-\gamma/\eta\} \cap \mathbb{Z}\,.$$

A *totally non-characteristic* integer $\kappa \in \mathbb{Z}$ is an integer such that $\kappa \not\equiv 0 \bmod p$, and $\mu(0,\kappa) \not\equiv 0 \bmod p$, that is,

$$\eta\kappa^2 + \gamma\kappa \not\equiv 0 \mod p\,.$$

We denote by $\mathcal{K}'$ the set of totally non-characteristic integers. For any $0 \neq \kappa \in \mathbb{Z}$ and any $\alpha \in R$, the *basic series* is

$$u_{E,\kappa,\alpha} := e_E\left(\int u^\mu_{a,\kappa,\alpha}\frac{dq}{q}\right) \in qK[[q]]\,, \tag{107}$$

where $e_E(T) \in TK[[q]][[T]]$ is the exponential of the formal group law $\mathcal{F}_E$, and $u^\mu_{a,\kappa,\alpha}$ is as in Equation 66. If in addition κ is characteristic (i.e. $\frac{\eta}{\gamma} = -\frac{1}{\kappa}$) then

$$u^\mu_{a,\kappa,\alpha} := \sum_{j\geq 0}(-1)^j\frac{\alpha^{\phi^j}}{F_j(p)}q^{\kappa p^j}\,. \tag{108}$$

Cf. Example 6.11.

Lemma 8.18. *Let us assume that $\kappa \in \mathbb{Z}\backslash p\mathbb{Z}$. Then we have $u_{E,\kappa,\alpha} \in qR[[q]]$. Moreover, the map $R \to R[[q]]$, $\alpha \mapsto u_{E,\kappa,\alpha}$ is a pseudo δ_p-polynomial map.*

Proof. In order to prove the first assertion, let

$$h := l_E(u_{E,\kappa,\alpha}) = \int u^\mu_{a,\kappa,\alpha}\frac{dq}{q}\,,$$

where $l_E(T) = l(T) \in K[[q]]$ is the logarithm of the formal group law $\mathcal{F}_E$. The isomorphism $\sigma(T) = T + \cdots \in R[[q]][[T]]$ between $\mathcal{F}_E$ and $\mathcal{F}_{\mathbb{G}_m}$ clearly has the property that

$$e_E(T) = \sigma(e_{\mathbb{G}_m}(T))\,,$$

where

$$e_{\mathbb{G}_m}(T) = \exp{(T)} - 1 = T + \frac{T^2}{2!} + \frac{T^3}{3!} + \cdots$$

So in order to check that $e_E(h) \in qR[[q]]$, it is enough to show that $e_{\mathbb{G}_m}(h) \in qR[[q]]$, that is to say, show that $\exp{(h)} \in 1 + qR[[q]]$. By Dwork's Lemma 2.4, it is enough to show that $(\phi - p)h \in pqR[[q]]$. Now, by Lemma 6.3 we have that

$$\begin{array}{rcl} p(\eta\kappa+\gamma)\alpha q^\kappa & = & p(\eta\delta_q - \gamma\phi_p + \gamma)u^\mu_{a,\kappa,\alpha} \\ & = & p(\eta\delta_q - \gamma\phi_p + \gamma)\delta_q h \\ & = & \delta_q(p\eta\delta_q - \gamma\phi_p + p\gamma)h\,. \end{array}$$

Hence

$$(p\eta\delta_q - \gamma\phi_p + p\gamma)h - p(\eta\kappa+\gamma)\alpha\kappa^{-1}q^\kappa \in R \cap qK[[q]] = 0\,, \tag{109}$$

which gives

$$\begin{array}{rcl}\gamma(\phi_p-p)h & = & p\eta\delta_q h - p(\eta\kappa+\gamma)\alpha\kappa^{-1}q^{\kappa}\\ & = & p\eta u^{\mu}_{a,\kappa,\alpha} - p(\eta\kappa+\gamma)\alpha\kappa^{-1}q^{\kappa}\\ & \in & pqR[[q]]\,,\end{array}$$

and we are done.

The second assertion is proved exactly as in Lemma 7.6. □

We have the following "diagonalization" result.

Lemma 8.19. *Let us assume that $\kappa \in \mathbb{Z}\backslash p\mathbb{Z}$, and that $\alpha \in R$. Then*

$$\begin{array}{rcl}\psi^2_q u_{E,\kappa,\alpha} & = & \gamma\kappa u^{\mu^{(p)}}_{a,\kappa,\alpha}\\ \psi^1_{pq} u_{E,\kappa,\alpha} & = & (\eta\kappa+\gamma)\alpha\kappa^{-1}q^{\kappa}\,.\end{array}$$

Proof. We have that

$$\begin{array}{rcl}\psi^2_q u_{E,\kappa,\alpha} & = & \gamma\delta^2_q l_E\left(e_E\left(\int u^{\mu}_{a,\kappa,\alpha}\frac{dq}{q}\right)\right)\\ & = & \gamma\delta^2_q\left(\int u^{\mu}_{a,\kappa,\alpha}\frac{dq}{q}\right)\\ & = & \gamma\delta_q u^{\mu}_{a,\kappa,\alpha}\\ & = & \gamma\kappa u^{\mu^{(p)}}_{a,\kappa,\alpha}.\end{array}$$

Consequently we have

$$\begin{array}{rcl}\delta^2_q\psi^1_{pq}u_{E,\kappa,\alpha} & = & \frac{1}{\gamma}(\eta\delta_q - p\gamma\phi_p+\gamma)\psi^2_q(u_{E,\kappa,\alpha})\\ & = & (\eta\delta_q - p\gamma\phi_p+\gamma)\kappa u^{\mu^{(p)}}_{a,\kappa,\alpha}\\ & = & \kappa\mu^{(p)}(\phi_p,\delta_q)u^{\mu^{(p)}}_{a,\kappa,\alpha}\\ & = & \kappa\mu^{(p)}(0,\kappa)\alpha q^{\kappa}\\ & = & \kappa(\eta\kappa+\gamma)\alpha q^{\kappa}\\ & = & \delta^2_q((\eta\kappa+\gamma)\alpha\kappa^{-1}q^{\kappa})\,.\end{array}$$

Hence

$$\delta_q(\psi^1_{pq}u_{E,\kappa,\alpha} - (\eta\kappa+\gamma)\alpha\kappa^{-1}q^{\kappa}) \in R\cap qR[[q]] = 0,$$

and, therefore,

$$\psi^1_{pq}u_{E,\kappa,\alpha} - (\eta\kappa+\gamma)\alpha\kappa^{-1}q^{\kappa} \in R\cap qR[[q]] = 0,$$

completing the proof. □

Remark 8.20.

1) For $\kappa \in \mathbb{Z}\backslash p\mathbb{Z}$ the map

$$\begin{array}{rcl}R & \to & \mathcal{E}(R[[q]])\\ \alpha & \mapsto & \iota(u_{E,\kappa,\alpha})\end{array}$$

is a group homomorphism.

2) In analogy with (31), if $\kappa \in \mathbb{Z}\backslash p\mathbb{Z}$, we may define *boundary value operators*

$$\begin{array}{rcl}B^0_{\kappa}:\mathcal{E}(qR[[q]]) & \to & R\,,\\ B^0_{\kappa}u & = & \Gamma_{\kappa}\psi^2_q u\,.\end{array}$$

Then, for $\kappa_1, \kappa_2 \in \mathbb{Z}\backslash p\mathbb{Z}$, we have

$$\begin{array}{rcl} B^0_{\kappa_1} u_{E,\kappa_2,\alpha} & = & \Gamma_{\kappa_1}\psi^2_q u_{E,\kappa_2,\alpha} \\ & = & \Gamma_{\kappa_1}\gamma\kappa_2 u^{\mu^{(p)}}_{a,\kappa_2,\alpha} \\ & = & \gamma\alpha\kappa_2\delta_{\kappa_1,\kappa_2}\,. \end{array}$$

3) For $\kappa \in \mathbb{Z}\backslash p\mathbb{Z}$ we have

(110) $$u_{E,\kappa,\zeta^\kappa\alpha}(q) = u_{E,\kappa,\alpha}(\zeta q)$$

for all $\zeta \in \mu(R)$. In particular, if $\alpha = \sum_{i=0}^\infty m_i\zeta_i^\kappa$, $\zeta_i \in \mu(R)$, $m_i \in \mathbb{Z}$, $v_p(m_i) \to \infty$, then

$$u_{E,\kappa,\alpha}(q) = \left[\sum_{i=0}^\infty\right][m_i](u_{E,1,1}(\zeta_i^\kappa q^\kappa)) = \left[\sum_{i=0}^\infty\right][m_i](u_{E,\kappa,1}(\zeta_i q))\,.$$

Here $[\sum]$ is the sum taken in the formal group law, and $[m_i](T) \in R[[q]][[T]]$ is the series induced by "multiplication by m_i" in the formal group. It is known that the sequence $[m_i](T)$ converges to 0 in $R[[q]][[T]]$ in the (p,T)-adic topology, so the series in the right hand side of the equality above converges in the (p,q)-adic topology of $R[[q]]$. Morally, $u_{E,\kappa,\alpha}$ is obtained from $u_{E,\kappa,1}$ via "convolution." (We have not defined "convolution" for groups not defined over R. This is possible but we will not do it here.)

4) We have the following "rationality" property: if $\alpha, \beta \in \mathbb{Z}_{(p)}$ then $u_{E,\kappa,\alpha} \in \mathbb{Z}_{(p)}[[q]]$ for $\kappa \in \mathbb{Z}\backslash p\mathbb{Z}$.

For the next Proposition, we denote by

(111) $$\mathcal{U}^1_{\rightarrow} := \{u \in E(A) \mid \psi^1_{pq} u = 0\}$$

the group of solutions of ψ^1_{pq} in $A = R((q))\widehat{\ }$, and we set

(112) $$\mathcal{U}^1_1 := \mathcal{U}^1_{\rightarrow} \cap E_1(A).$$

Note that there is no natural analogue of $\mathcal{U}_{\leftarrow}$ in this case.

We let $E_0(A)$ be the subgroup of all $u \in E(A)$ that are solutions to all δ_q-characters of E. By [22] or [4], $E_0(A)$ is actually the set of all solutions of the Manin map ψ^2_q. We set

(113) $$\mathcal{U}^1_0 := \mathcal{U}^1_{\rightarrow} \cap E_0(A).$$

We should view $\mathcal{U}^1_0$ as a substitute for the groups of stationary solutions in the cases of $\mathbb{G}_a$ and $\mathbb{G}_m$, respectively.

Theorem 8.21. *Let $E_{\beta q}$ be the Tate curve with parameter βq, where $\beta \in R^\times$. Let $\mathcal{U}^1_{\rightarrow}$ be the group of solutions in $R((q))\widehat{\ }$ of the $\{\delta_p, \delta_q\}$-character ψ^1_{pq}. Then the following hold:*

1) *If $\frac{\eta}{\gamma} = -\frac{1}{\kappa}$ for some integer $\kappa \geq 1$ coprime to p, and if $u_{E,\kappa,\alpha}$ is the basic series in* (107)*, then*

$$\begin{array}{rcl} \mathcal{U}^1_{\rightarrow} & = & \mathcal{U}^1_0 + \mathcal{U}^1_1\,, \\ \mathcal{U}^1_1 & = & \{u_{E,\kappa,\alpha} \mid \alpha \in R\}\,. \end{array}$$

2) *If $\frac{\eta}{\gamma}$ is not of the form $\frac{\eta}{\gamma} = -\frac{1}{\kappa}$ for some integer $\kappa \geq 1$ coprime to p, then*

$$\mathcal{U}^1_1 = 0\,.$$

Proof. It is sufficient to prove the following claims:

1) If $\frac{\eta}{\gamma} = -\frac{1}{\kappa}$ for some integer $\kappa \geq 1$ coprime to p, then $\psi^1_{pq} u_{E,\kappa,\alpha} = 0$ for all α.
2) Assume $\psi^1_{pq} u = 0$ for some $0 \neq u \in qR[[q]]$. Then $\frac{\eta}{\gamma} = -\frac{1}{\kappa}$ for some integer $\kappa \geq 1$ coprime to p, and $u = u_{E,\kappa,\alpha}$ for some α.
3) If $\frac{\eta}{\gamma} = -\frac{1}{\kappa}$ for some integer $\kappa \geq 1$ coprime to p, and if $u \in E(A)$ is such that $\psi^1_{pq} u = 0$, then there exists α_u such that $\psi^2_q (u - u_{E,\kappa,\alpha_u}) = 0$.

The first of these claims follows directly by Lemma 8.19.

For the second claim, we write

$$0 = \delta^2_q \psi^1_{pq} u = (\eta \delta_q - p\gamma\phi_p + \gamma)\, \delta^2_q (l_E(u))\,.$$

Now

$$0 \neq v := \delta^2_q (l_E(u)) \in qR[[q]]\,,$$

so, by Theorem 6.10, there exists an integer κ such that $\eta\kappa + \gamma = 0$ and $v = u^{\mu^{(p)}}_{a,\kappa,\alpha}$ for some $\alpha \in R$. By (70), $\delta_q l_E(u) = u^{\mu}_{a,\kappa,\alpha}$ so

$$u = e_E \left(\int u^{\mu}_{a,\kappa,\alpha} \frac{dq}{q} \right) = u_{E,\kappa,\alpha}\,,$$

and the result follows.

For the third claim note that, as before, we have

$$0 = \delta^2_q \psi^1_{pq} u = \frac{1}{\gamma} (\eta \delta_q - p\gamma\phi_p + \gamma) \psi^2_q u\,.$$

By Theorem 6.10, $\psi^2_q u = a_0 + u^{\mu^{(p)}}_{a,\kappa,\alpha}$ for some $\alpha \in R$ and some $a_0 \in R$, with $(\eta\delta_q - p\gamma\phi_p + \gamma) a_0 = 0$. The latter simply says $p\phi_p(a_0) = a_0$, and since $v_p(\phi_p(a_0)) = v_p(a_0)$, we must have $a_0 = 0$. Hence $\psi^2_q u = u^{\mu^{(p)}}_{a,\kappa,\alpha}$. On the other hand,

$$\psi^2_q u_{E,\kappa,\alpha/\gamma\kappa} = \gamma\kappa u^{\mu^{(p)}}_{a,\kappa,\alpha/\gamma\kappa} = u^{\mu^{(p)}}_{a,\kappa,\alpha}.$$

and so $\psi^2_q (u - u_{E,\kappa,\alpha/\gamma\kappa}) = 0$, completing the proof. □

Remark 8.22. The condition in Theorem 8.21 that the quotient of the Kodaira-Spencer classes has the form

$$\frac{\eta}{\gamma} = -\frac{1}{\kappa}$$

for some integer $\kappa \geq 1$ coprime to p has a nice interpretation in terms of wave lengths (in the sense of Remark 7.10). The condition is equivalent to saying that the wave length of the parameter βq of the Tate curve belongs to

$$\left\{ -\frac{c}{\gamma}, -\frac{2c}{\gamma}, -\frac{3c}{\gamma}, \ldots \right\} \cap \mathbb{Z}_p^\times\,.$$

This is, again, a "quantization" condition.

Corollary 8.23. *Under the hypotheses of Theorem* 8.21 *and the assumption that* $\frac{\eta}{\gamma} = -\frac{1}{\kappa}$ *for some integer* κ, *let* $u \in \mathcal{U}^1_1$. *Then we have that the series* $\overline{\psi^2_q u} \in k[[q]]$ *is integral over* $k[q]$, *and the field extension* $k(q) \subset k(q, \overline{\psi^2_q u})$ *is Abelian with Galois group killed by* p.

Proof. This follows immediately by Theorem 8.21 and Lemmas 6.6 and 8.19, respectively. □

The next Corollary says that the "boundary value problem at $q = 0$" is well posed.

Corollary 8.24. *Under the hypotheses of Theorem* 8.21 *and the assumption that* $\frac{\eta}{\gamma} = -\frac{1}{\kappa}$ *for some integer* κ*, then for any* $\alpha \in R$ *there exists a unique* $u \in \mathcal{U}_1^1$ *such that* $B_\kappa^0 u = \alpha$.

Next we study $\{\delta_p, \delta_q\}$-characters of order 2 of the form $\psi_q^2 + \lambda\psi_p^2$.

Theorem 8.25. *Let* $E_{\beta q}$ *be the Tate curve with parameter* βq *for* β *not a root of unity. Let* $\mathcal{U}^2_{\rightarrow}$ *be the set of solutions of the* $\{\delta_p, \delta_q\}$*-character* $\psi_q^2 + \lambda\psi_p^2$ *in* $A = R((q))\widehat{\ }$*, where* $\lambda \in R$*. We set*

$$\begin{array}{rcl} \mathcal{U}_0^2 & := & \mathcal{U}^2_{\rightarrow} \cap E_0(A)\,, \\ \mathcal{U}_1^2 & := & \mathcal{U}^2_{\rightarrow} \cap E_1(A)\,. \end{array}$$

Then the following hold:

1) *If* $\frac{\eta}{\gamma} = -\frac{1}{\kappa}$ *and* $\lambda = \kappa^3$ *for some integer* $\kappa \geq 1$ *coprime to* p*, and if* $u_{E,\kappa,\alpha}$ *is the basic series of* ψ_{pq}^1 *(cf. (107)), then*

$$\begin{array}{rcl} \mathcal{U}^2_{\rightarrow} & = & \mathcal{U}_0^2 + \mathcal{U}_1^2\,, \\ \mathcal{U}_1^2 & = & \{u_{E,\kappa,\alpha} \mid \alpha \in R\}\,. \end{array}$$

2) *If* $\frac{\lambda\eta^\phi}{\gamma} \notin \{-n^2 \mid n \in \mathbb{Z}\}$*, then* $\mathcal{U}_1^2 = 0$.

Proof. We prove assertion 1). The fact that $\mathcal{U}_1^2$ consists of exactly the $u_{E,\kappa,\alpha}$'s follows directly by Theorem 8.21 and Proposition 8.14. We just need to observe that, by Theorem 6.10, the map

$$(\gamma\eta^2\delta_q + \gamma^2\eta\phi_p - \gamma^2\eta) : qR[[q]] \to qR[[q]]$$

is injective. Now if $u \in \mathcal{U}^2_{\rightarrow}$ then, by Proposition 8.14, we have

$$(\gamma\eta^2\delta_q + \gamma^2\eta\phi_p - \gamma^2\eta)\psi_{pq}^1 u = 0\,.$$

By Theorem 6.10, $\psi_{pq}^1 u \in R$. Again, by Proposition 8.14, we get

$$0 = \gamma\delta_q^2\psi_{pq}^1 u = (\eta\delta_q - p\gamma\phi_p + \gamma)\psi_q^2 u\,.$$

By Theorem 6.10 there is an $\alpha \in R$ such that

$$\psi_q^2 u = u^{\mu^{(p)}}_{a,\kappa,\alpha}\,.$$

On the other hand, as in the proof of Theorem 8.21, we have that $u^{\mu^{(p)}}_{a,\kappa,\alpha} = \psi_q^2 u_{E,\kappa,\alpha/\gamma\kappa}$. Hence $\psi_q^2(u - u_{E,\kappa,\alpha/\gamma\kappa}) = 0$, and the assertion is proved.

In order to prove assertion 2), let us assume $u \in qR[[q]]$ is such that $(\psi_q^2+\lambda\psi_p^2)u = 0$. Then

$$\begin{array}{rcl} 0 = \delta_q(\psi_q^2 + \lambda\psi_p^2)u & = & \delta_q\left[\gamma\delta_q^2 + \frac{\lambda}{p}\left(\eta\phi_p^2 - (\eta^\phi + p\eta)\phi_p + p\eta^\phi\right)\right] l(u) \\ & = & [\gamma\delta_q^2 + p\lambda\eta\phi_p^2 - \lambda(\eta^\phi + p\eta)\phi_p + \lambda\eta^\phi]\delta_q l(u)\,. \end{array}$$

By our assumption, $\frac{\lambda\eta^\phi}{\gamma} + n^2 \neq 0$ for all $n \in \mathbb{Z}$. By Theorem 6.10,

$$\gamma\delta_q^2 + p\lambda\eta\phi_p^2 - \lambda(\eta^\phi + p\eta)\phi_p + \lambda\eta^\phi$$

has no non-zero solution in $qR[[q]]$. Since $\delta_q l(u) \in qR[[q]]$, we get $\delta_q l(u) = 0$ so $l(u) \in K \cap qK[[q]] = 0$ so $u = 0$. ∎

Next we address the boundary value problem at $q \neq 0$ for ψ^1_{pq}. For $q_0 \in pR$ and $\beta \in R^\times$ not a root of unity, we let $\mathcal{E}_{\beta q_0}$ be the (non-smooth) plane projective curve that in the affine plane is given by the equation

$$y^2 = x^3 - \frac{1}{48}E_4(\beta q_0)x + \frac{1}{864}E_6(\beta q_0)\,.$$

There is a commutative diagram of groups

$$\begin{array}{ccc} qR[[q]] & \stackrel{\iota}{\to} & \mathcal{E}_{\beta q}(R[[q]]) \\ \downarrow & & \downarrow \pi_{q_0} \\ pR & \stackrel{\iota_{q_0}}{\to} & \mathcal{E}_{\beta q_0}(R) \end{array} \,, \tag{114}$$

where the vertical arrows are induced by the ring homomorphism

$$\begin{array}{ccc} R[[q]] & \to & R \\ u(q) & \mapsto & u(q_0) \end{array}\,,$$

ι_{q_0} sends pa into $(pa, (pa)^3 - \frac{1}{48}E_4(\beta q_0)(pa)^7 + \cdots)$ for $a \in R$, and the image of ι_{q_0} is $\mathcal{E}_{\beta q_0}(pR)$, the preimage of 0 via the reduction modulo p mapping

$$\mathcal{E}_{\beta q_0}(R) \to \mathcal{E}_{\beta q_0}(k)\,. \tag{115}$$

So we have $\iota_{q_0}(pR) = \mathcal{E}_{\beta q_0}(pR)$. We note that we have usually identified $\iota(qR[[q]])$ with $qR[[q]]$, but in order to avoid the potential confusion produced by the choice of q_0, we will **not** identify $\iota_{q_0}(pR)$ with pR.

Let $\mathcal{E}'_{\beta q_0}(R)$ be the pull-back by the map (115) of the locus $\mathcal{E}^{reg}_{\beta q_0}(k)$ of regular points on the cubic $\mathcal{E}_{\beta q_0}(k)$. So we have inclusions of groups

$$\mathcal{E}_{\beta q_0}(pR) \subset \mathcal{E}'_{\beta q_0}(R) \subset \mathcal{E}_{\beta q_0}(R)\,.$$

Let us recall some facts about torsion points on Tate curves. There is a natural injective homomorphism

$$\begin{array}{rcl} \tau : \mu(R) & \to & \mathcal{E}_{\beta q}(R[[q]]) \\ v & \mapsto & (X(q,v) + \frac{1}{12}, -Y(q,v) - \frac{1}{2}X(q,v))\,, \quad v \neq 1\,, \\ 1 & \mapsto & \infty\,, \end{array} \tag{116}$$

given by the series in (106). (The formula makes sense because if $1 \neq v \in \mu(R)$ then $1-v \in R^\times$.) The composition of the homomorphism in (116) with the specialization map π_{q_0} in (114) gives a homomorphism

$$\tau_{q_0} : \mu(R) \to \mathcal{E}_{\beta q_0}(R)$$

The composition of τ_{q_0} with the reduction mod p mapping $\mathcal{E}_{\beta q_0}(R) \to \mathcal{E}_{\beta q_0}(k)$ is the map

$$\begin{array}{rcl} \mu(R) & \to & \mathcal{E}_{\beta q_0}(k) \\ \zeta & \mapsto & \left(\frac{\zeta}{(1-\zeta)^2} + \frac{1}{12} \bmod p, -\frac{\zeta^2}{(1-\zeta)^3} - \frac{\zeta}{2(1-\zeta)^2} \bmod p\right)\,, \ \zeta \neq 1\,, \\ 1 & \mapsto & \infty\,, \end{array}$$

which is an isomorphism of $\mu(R) \simeq k^\times$ onto $\mathcal{E}^{reg}_{\beta q_0}(k)$. We conclude that the mapping τ_{q_0} above is an injective map, and any point $P \in \mathcal{E}'_{\beta q_0}(R)$ can be written uniquely as

$$P = \iota_{q_0}(pa) + \tau_{q_0}(\zeta)\,,$$

where $a \in R$ and $\zeta \in \mu(R)$.

Corollary 8.26. *Let $\mathcal{U}^1_{\to}$ be the set of solutions of ψ^1_{pq} in $R((q))\widehat{\ }$, and let*

$$\mathcal{U}'_{\to} := \tau(\mu(R)) \cdot \mathcal{U}^1_1 \subset \mathcal{U}^1_{\to}\,.$$

Assume $\frac{\eta}{\gamma} = -\frac{1}{\kappa}$ with $\kappa \geq 1$ an integer coprime to p. Then the following hold:

1) *For any $q_0 \in p^\nu R^\times$ with $\nu \geq 1$ and any $g \in \iota_{q_0}(p^{\kappa\nu}R) \subset \iota_{q_0}(pR) = \mathcal{E}_{\beta q_0}(pR)$, there exists a unique $u \in \mathcal{U}^1_1 = \iota(qR[[q]]) = qR[[q]]$ such that $u(q_0) = g$.*
2) *Assume $\kappa = 1$. Then for any $q_0 \in pR^\times$ and any $g \in \mathcal{E}'_{\beta q_0}(R)$, there exists a unique $u \in \mathcal{U}'_{\to}$ such that $u(q_0) = g$.*

Proof. Let us prove assertion 1). By Theorem 8.21 and Equations (107) and (108), it is enough to show that the mapping $R \to p^{\kappa\nu}R$ defined by

$$\alpha \mapsto e_E\left(\sum_{j\geq 0}(-1)^j \frac{\alpha^{\phi^j}}{\kappa p^j F_j(p)} q_0^{\kappa p^j}\right)$$

is a bijection. Let us recall that $e_E : p^N R \to \iota_{q_0}(p^N R)$ is an isomorphism for all N. So it is enough to show that the map $R \to p^{\kappa\nu}R$ given by

$$\alpha \mapsto \sum_{j\geq 0}(-1)^j \frac{\alpha^{\phi^j}}{\kappa p^j F_j(p)} q_0^{\kappa p^j}\,,$$

is a bijection. But this is clear by Lemma 6.15.

For the proof of assertion 2), we let $g \in \mathcal{E}'_{\beta q_0}(R)$, and write $g = \tau_{q_0}(\zeta_0) + w_0$, $w_0 \in \mathcal{E}_{\beta q_0}(pR)$. By assertion 1), there exists $w \in \mathcal{U}^1_1$ such that $w(q_0) = w_0$. Let us set $u := \tau(\zeta_0) + w$. Then $u(q_0) = \tau_{q_0}(\zeta_0) + w_0 = g$, which completes the proof of the existence part of the assertion. In order to prove uniqueness, we let $u_1 = \tau(\zeta_1) + w_1$ with $w_1 \in \mathcal{U}^1_1$ such that $u_1(q_0) = g$. Then

$$\tau_{q_0}(\zeta_1) + w_1(q_0) = \tau_{q_0}(\zeta_0) + w_0\,.$$

This implies that $\zeta_1 = \zeta_0$ and $w_1(q_0) = w_0$. By the uniqueness in assertion 1), we get $w_1 = w$. We conclude that $u_1 = u$, and we are done. $\square$

The following Corollary is concerned with the inhomogeneous equation $\psi^1_{pq}u = \varphi$, and it is an immediate consequence of Lemmas 8.19, 8.24, and 6.6

Corollary 8.27. *Under the hypotheses of Theorem 8.21, assume that $\frac{\eta}{\gamma} = -\frac{1}{\kappa}$ for some integer κ, and that $\varphi \in qR[[q]]$ is a series whose support is contained in the set $\mathcal{K}'$ of totally non-characteristic integers of ψ^1_{pq}. Then the equation $\psi^1_{pq}u = \varphi$ has a unique solution $u \in E_1(A)$ such that the support of $\psi^2_q u$ does not contain κ. Moreover, if $\bar{\varphi} \in k[q]$ the series $\overline{\psi^2_q u} \in k[[q]]$ is integral over $k[q]$ and the field extension $k(q) \subset k(q, \overline{\psi^2_q u})$ is Abelian with Galois group killed by p.*

Remark 8.28. Let us assume that $\frac{\eta}{\gamma} = -1$. By Corollary 8.26, for any $q_0 \in pR^\times$ the group homomorphism

$$\begin{array}{rcl} S_{q_0} : \mu(R) \times R & \to & \mathcal{E}'_{\beta q_0}(R) \\ (\xi, \alpha) & \mapsto & \tau_{q_0}(\xi) + \iota_{q_0}(u_{E,1,\alpha}(q_0)) \end{array}$$

is an isomorphism. So, for any $q_1, q_2 \in pR^\times$ we have a group isomorphism

$$S_{q_1,q_2} := S_{q_2} \circ S^{-1}_{q_1} : \mathcal{E}'_{\beta q_1}(R) \to \mathcal{E}'_{\beta q_2}(R)\,.$$

The latter mapping can be viewed as the "propagator" attached to ψ^1_{pq}.

It is natural to make the propagator act as an endomorphism of a given group. We accomplish this by defining a new *propagator*

$$S^{q_0}_{q_1,q_2} := \Gamma_{q_0} \circ \Gamma^{-1}_{q_2} \circ S_{q_1,q_2} \circ \Gamma_{q_1} \circ \Gamma^{-1}_{q_0} : \mathcal{E}'_{\beta q_0}(R) \to \mathcal{E}'_{\beta q_0}(R)\,,$$

where Γ_{q_i} is the group isomorphism

$$\begin{array}{ccc} \mu(R) \times R & \stackrel{\Gamma_{q_i}}{\to} & \mathcal{E}'_{\beta q_i}(R) \\ (\xi, a) & \mapsto & \iota_{q_i}(pa) + \tau_{q_i}(\xi) \end{array} .$$

Then it is easy to see that for $\zeta_1, \zeta_2 \in \mu(R)$ we have that

$$S^{q_0}_{q_0,\zeta_1\zeta_2 q_0} = S^{q_0}_{q_0,\zeta_2 q_0} \circ S^{q_0}_{q_0,\zeta_1 q_0}\,,$$

which, once again, we view as a (weak) form of the "Huygens principle."

8.3. **Elliptic curves over** R**.** In this section we consider an elliptic curve E over $A := R$ defined by

$$f = y^2 - (x^3 + a_4 x + a_6)\,,$$

where $a_4, a_6 \in R$, and equip it with the 1-form

$$\omega = \frac{dx}{y}\,.$$

We use the notation and discussion in Example 2.16 that applies to E over R.

Clearly, $f^1_q = 0$. Let us fix in what follows the étale coordinate $T = -\frac{x}{y}$, and denote by $l(T) = l_E(T) \in K[[T]]$ the logarithm of the formal group $\mathcal{F}_E$ of E with respect to T.

Proposition 8.29. *The following holds:*

1) *The image of* ψ^1_q *in*
$$\mathcal{O}\left(J^1_{pq}\left(\frac{A[x,y]}{(f)}\right)\right) = \frac{A[x,y,\delta_p x,\delta_p y,\delta_q x,\delta_q y]\hat{}}{(f,\delta_p f,\delta_q f)}$$
is equal to $\dfrac{\delta_q x}{y}$.

2) *The image of* ψ^1_q *in* $A[[T]][\delta_p T, \delta_q T]\hat{}$ *is equal to* $\delta_q l(T) = \dfrac{dl}{dT}(T) \cdot \delta_q T$.

The first assertion above says that the $\{\delta_p, \delta_q\}$-character ψ^1_q attached to (E, ω) coincides with the *Kolchin's logarithmic derivative* [20].

Proof. We first check that $\frac{\delta_q x}{y}$ comes from a $\{\delta_p, \delta_q\}$-character. For that, it is enough to show that its image $\left(\frac{\delta_q x}{y}\right)(T, \delta_q T)$ in $A[[T]][\delta_p T, \delta_q T]\hat{}$ defines a homomorphism from the formal group of $J^1_{pq}(E)$ to the formal group of $\mathbb{G}_a$. This is so because "a partially defined map, which generically is a homomorphism, is an everywhere defined homomorphism." We now recall that ω and l are related by the equation

$$\omega = \frac{dl}{dT}(T) \cdot dT\,.$$

Consequently, if we denote by $x(T)$ and $y(T)$ the images of x and y in $R((T))$, respectively, we have that

$$\frac{dl}{dT}(T) \cdot dT = \left(\frac{dx}{y}\right)(T) = (y(T))^{-1}\frac{dx}{dT}(T) \cdot dT\,.$$

Hence

$$\left(\frac{\delta_q x}{y}\right)(T, \delta_q T) = (y(T))^{-1}\frac{dx}{dT}(T)\cdot \delta_q T = \frac{dl}{dT}(T)\cdot \delta_q T = \delta_q l(T)\,. \tag{117}$$

But clearly $\delta_q l(T)$ defines a homomorphism at the level of formal groups. Hence $\frac{\delta_q x}{y}$ is a $\{\delta_p, \delta_q\}$-character of E. Now, (117) also shows that

$$\psi^1_q - \frac{\delta_q x}{y} \in A[[T]]\,,$$

and thus, $\psi^1_q - \frac{\delta_q x}{y}$ defines a homomorphism $\hat{E} \to \hat{\mathbb{G}}_a$. Therefore, $\psi^1_q - \frac{\delta_q x}{y} = 0$ and assertion 1) is proved.

The second assertion follows from the first in combination with (117). $\square$

Remark 8.30. As in the previous subsection, we identify $q^{\pm 1}R[[q^{\pm 1}]]$ with its image $E(q^{\pm 1}R[[q^{\pm 1}]])$ in $E(R[[q^{\pm 1}]])$ under the embedding

$$\iota : q^{\pm 1}R[[q^{\pm 1}]] \to E(q^{\pm 1}R[[q^{\pm 1}]])$$

given in (T, W)- coordinates by

$$u \mapsto (u, u^3 + a_4 u^7 + \cdots)\,.$$

In the sequel, we fix an elliptic curve E/R. We define next the *characteristic polynomial* of a $\{\delta_p, \delta_q\}$-character, *non-degenerate* $\{\delta_p, \delta_q\}$-characters, their *characteristic integers*, and *basic series*. We distinguish the two cases $f^1_p \neq 0$ and $f^1_p = 0$, respectively.

If $f^1_p \neq 0$, by [9], p. 197, $v_p(f^2_p) \geq v_p(f^1_p)$. We may therefore consider the $\{\delta_p, \delta_q\}$-character

$$\psi^2_{p,nor} := \frac{1}{f^1_p}\psi^2_p \in \mathbf{X}^2_{pq}(E)\,.$$

Its image in $R[[T]][\delta_p T, \delta_p^2 T]\hat{}$ is

$$\psi^2_{p,nor} = \frac{1}{p}(\phi_p^2 + \gamma_1 \phi_p + p\gamma_0)l(T),$$

where

$$\gamma_1 := -\frac{f^2_p}{f^1_p} \in R, \quad \gamma_0 := \frac{(f^1_p)^{\phi}}{f^1_p} \in R^{\times}.$$

By Proposition 8.8, any $\{\delta_p, \delta_q\}$-character of E is a K-multiple of a $\{\delta_p, \delta_q\}$-character of the form

$$\psi_E := \nu(\phi_p, \delta_q)\psi^1_q + \lambda(\phi_p)\psi^2_{p,nor}\,, \tag{118}$$

where $\nu(\xi_p, \xi_q) \in R[\xi_p, \xi_q]$ and $\lambda(\xi_p) \in R[\xi_p]$. The Picard-Fuchs symbol of ψ_E with respect to T is easily easily seen to be

$$\sigma(\xi_p, \xi_q) = p\nu(\xi_p, \xi_q)\xi_q + \lambda(\xi_p)(\xi_p^2 + \gamma_1 \xi_p + p\gamma_0)\,.$$

Hence, the Fréchet symbol of ψ_E with respect to ω is

$$\theta(\xi_p, \xi_q) = \frac{\sigma(p\xi_p, \xi_q)}{p} = \nu(p\xi_p, \xi_q)\xi_q + \lambda(p\xi_p)(p\xi_p^2 + \gamma_1 \xi_p + \gamma_0)\,.$$

Definition 8.31. Let us assume that E/R has $f^1_p \neq 0$, and let ψ_E be a $\{\delta_p, \delta_q\}$-character of E of the form in (118). We define the *characteristic polynomial* $\mu(\xi_p, \xi_q)$

of ψ_E to be the Fréchet symbol $\theta(\xi_p, \xi_q)$ of ψ_E with respect to ω. We say that the $\{\delta_p, \delta_q\}$-character ψ_E is *non-degenerate* if $\mu(0,0) \in R^\times$, or equivalently, if $\lambda(0) \in R^\times$. For a non-degenerate character ψ_E, we define its *characteristic integers* to be the integers κ such that $\mu(0,\kappa) = 0$ (so any such κ is coprime to p), and denote by $\mathcal{K}$ the set of all such. We call κ a *totally non-characteristic* integer if $\kappa \not\equiv 0$ mod p and $\mu(0,\kappa) \not\equiv 0$ mod p. The set of all totally non-characteristic integers is denoted by $\mathcal{K}'$. For $0 \neq \kappa \in \mathbb{Z}$ and $\alpha \in R$, we define the *basic series* by

$$u_{E,\kappa,\alpha} = e_E\left(\int u_{a,\kappa,\alpha}\frac{dq}{q}\right) \in q^{\pm 1}K[[q^{\pm 1}]]\,, \tag{119}$$

where

$$u_{a,\kappa,\alpha} := u^{\mu}_{a,\kappa,\alpha}; \tag{120}$$

Cf. (66).

Let us now consider now the case where $f^1_p = 0$. We may then look at the $\{\delta_p, \delta_q\}$-character $\psi^1_p \in \mathbf{X}^1_p(E)$, cf. (101). Its image in $R[[T]][\delta_p T]\hat{}$ is

$$\psi^1_p = \frac{1}{p}(\phi_p + p\gamma_0)l(T)\,,$$

where $\gamma_0 \in R$; cf. [9], Remark 7.21. (We actually have $\gamma_0 \in R^\times$ if the cubic defining E has coefficients in $\mathbb{Z}_p$; cf. [6], Theorem 1.10.) By Proposition 8.9, any $\{\delta_p, \delta_q\}$-character of E is a K-multiple of a $\{\delta_p, \delta_q\}$-character of the form

$$\psi_E := \nu(\phi_p, \delta_q)\psi^1_q + \lambda(\phi_p)\psi^1_p\,, \tag{121}$$

where $\nu(\xi_p, \xi_q) \in R[\xi_p, \xi_q]$ and $\lambda(\xi_p) \in R[\xi_p]$. We easily see that the Picard-Fuchs symbol of ψ_E is given by

$$\sigma(\xi_p, \xi_q) = p\nu(\xi_p, \xi_q)\xi_q + \lambda(\xi_p)(\xi_p + p\gamma_0)\,.$$

Thus, the Fréchet symbol at the origin (with respect to dT) is

$$\theta(\xi_p, \xi_q) = \frac{\sigma(p\xi_p, \xi_q)}{p} = \nu(p\xi_p, \xi_q)\xi_q + \lambda(p\xi_p)(\xi_p + \gamma_0)\,.$$

Definition 8.32. Let us assume that E/R has $f^1_p = 0$. (Recall that then E/R has ordinary reduction mod p; cf. [9], Corollary 8.89.), and let us fix a $\{\delta_p, \delta_q\}$-character of E of the form ψ_E as in (121). We define the *characteristic polynomial* $\mu(\xi_p, \xi_q)$ of ψ_E to be the Fréchet symbol $\theta(\xi_p, \xi_q)$ of ψ_E with respect to ω. We say that the $\{\delta_p, \delta_q\}$-character ψ_E is *non-degenerate* if $\mu(0,0) \in R^\times$, or equivalently, if $\lambda(0) \in R^\times$ and $\gamma_0 \in R^\times$. For a non-degenerate character ψ_E, we define the *characteristic integers* to be the integers κ such that $\mu(0,\kappa) = 0$ (so any such κ is coprime to p), and denote by $\mathcal{K}$ the set of all such. We say that κ is a *totally non-characteristic* integers is $\kappa \not\equiv 0$ mod p and $\mu(0,\kappa) \not\equiv 0$ mod p. The set of all totally non-characteristic integers is denoted by $\mathcal{K}'$. For $0 \neq \kappa \in \mathbb{Z}$ and $\alpha \in R$, we define the *basic series*

$$u_{E,\kappa,\alpha} = e_E\left(\int u_{a,\kappa,\alpha}\frac{dq}{q}\right) \in q^{\pm 1}K[[q^{\pm 1}]]\,, \tag{122}$$

where

$$u_{a,\kappa,\alpha} := u^{\mu}_{a,\kappa,\alpha}; \tag{123}$$

cf. (66).

In the sequel, we consder an elliptic curve E/R without imposing any a priori restriction on the vanishing of f^1_p.

Example 8.33. Let us assume that $\nu = 1$, $\lambda \in R^\times$, that is to say, ψ_E is either $\psi^1_q + \lambda\psi^2_{p,nor}$ or $\psi^1_q + \lambda\psi^1_p$ according as $f^1_p \neq 0$ or $f^1_p = 0$, respectively. Then the characteristic polynomial is unmixed (provided that $\gamma_1 \in R^\times$ in case $f^1_p \neq 0$),

$$\mathcal{K} = \{-\lambda\gamma_0\} \cap \mathbb{Z}\,,$$

and $\mathcal{K} \neq \emptyset$ if, and only if, $\lambda\gamma_0 \in \mathbb{Z}$. In this case, ψ_E should be viewed as an analogue of either the heat equation or the convection equation according as $f^1_p \neq 0$ or $f^1_p = 0$, respectively.

Example 8.34. Let us now assume that $\nu = \xi_q$ and $\lambda \in R^\times$, that is to say, ψ_E is either $\delta_q\psi^1_q + \lambda\psi^2_{p,nor}$ or $\delta_q\psi^1_q + \lambda\psi^1_p$ according as $f^1_p \neq 0$ or $f^1_p = 0$, respectively. Then the characteristic polynomial is unmixed (provided that $\gamma_1 \in R^\times$ in case $f^1_p \neq 0$),

$$\mathcal{K} = \{\pm\sqrt{-\lambda\gamma_0}\} \cap \mathbb{Z}\,,$$

and $\mathcal{K} \neq \emptyset$ if, and only if, $-\lambda\gamma_0$ is a perfect square. If this the case, ψ_E should be viewed as analogue of either the wave equation or the sideways heat equation according as $f^1_p \neq 0$ or $f^1_p = 0$, respectively.

Example 8.35. Let us consider the "simplest" of the energy functions in the case where $f^1_p = 0$:

$$H = a(\psi^1_q)^2 + 2b\psi^1_p\psi^1_q + c(\psi^1_p)^2\,.$$

Here, $a, b, c \in R$.

A computation essentially identical to the one in Example 7.5 leads to the following formula for the Euler-Lagrange equation $\epsilon^1_{H,\partial}$ attached to H and the vector field ∂ dual to ω:

$$\epsilon^1_{H,\partial} = (-2a^\phi\phi_p\delta_q - 2b^\phi\phi_p^2 + 2b)\psi^1_q + (2c^\phi\gamma_0^\phi\phi_p + 2c)\psi^1_p\,.$$

Hence, the characteristic polynomial of $\epsilon^1_{H,\partial}$ is

$$\mu(\xi_p,\xi_q) = (-2a^\phi p\xi_p\xi_q - 2b^\phi p^2\xi_p^2 + 2b)\xi_q + (2c^\phi\gamma_0^\phi p\xi_p + 2c)(\xi_p + \gamma_0)\,,$$

so the $\{\delta_p,\delta_q\}$-character $\epsilon^1_{H,\partial}$ is non-degenerate if, and only if, $c \in R^\times$ and $\gamma_0 \in R^\times$. Moreover, the set of characteristic integers is

$$\mathcal{K} = \{-\frac{c\gamma_0}{b}\} \cap \mathbb{Z}\,.$$

On the other hand, when $f^1_p \neq 0$ we consider the "simplest" energy function

$$H = a(\psi^1_q)^2 + 2b\psi^1_q\psi^2_{p,nor} + c(\psi^2_{p,nor})^2\,,$$

for $a, b, c \in R$. Then we have the following values for the Fréchet symbols:

$$\begin{array}{rcl} \theta_{\psi^1_q,\omega} & = & \xi_q \\ \theta_{\psi^2_{p,nor},\omega} & = & p\xi_p^2 + \gamma_1\xi_p + \gamma_0 \end{array}\,,$$

and we get the following formula for the Euler-Lagrange equation attached to H and the vector field ∂ dual to ω:

$$\begin{array}{rcl} \epsilon^2_{H,\partial} & = & (-2b^{\phi^2}p\phi_p^3 - 2a^{\phi^2}\phi_p^2\delta_q)\psi^1_q \\ & & +[2b^{\phi^2}(\gamma_0^{\phi^2} - \gamma_1^\phi)\phi_p^2 + (2b^\phi\gamma_1^\phi - 2b^{\phi^2}\gamma_0^\phi)\phi_p + 2bp]\psi^1_q \\ & & +(2c^{\phi^2}\gamma_0^{\phi^2}\phi_p^2 + 2c^\phi\gamma_1^\phi\phi_p + 2cp)\psi^2_{p,nor}\,. \end{array}$$

In particular $\epsilon^2_{H,\partial}$ is degenerate for all values of a, b, c. More is true, actually: $\epsilon^2_{H,\partial}$ is not a K-multiple of a non-degenerate $\{\delta_p, \delta_q\}$-character.

Lemma 8.36. *Let $\kappa \in \mathbb{Z}\backslash p\mathbb{Z}$. Then we have $u_{E,\kappa,\alpha} \in q^{\pm 1}R[[q^{\pm 1}]]$ for all $\alpha \in R$, and*

$$\begin{array}{ccc} R & \to & R[[q^{\pm 1}]] \\ \alpha & \mapsto & u_{E,\kappa,\alpha} \end{array}$$

is a pseudo δ_p-polynomial mapping. Consequently,

$$\begin{array}{ccc} R & \to & E(R[[q^{\pm 1}]]) \\ \alpha & \mapsto & \iota(u_{E,\kappa,\alpha}) \end{array}$$

is also a pseudo δ_p-polynomial mapping.

Proof. As an element of $R[[T]][\delta_p T, \delta_q T, \delta_p^2 T, \delta_p\delta_q T, \delta_q^2 T]\widehat{\ }$, the $\{\delta_p, \delta_q\}$-character ψ_E coincides with

$$\psi_E = \left(\nu(\phi_p, \delta_q)\delta_q + \frac{\lambda(\phi_p)}{p}\phi_p^2 + \frac{\lambda(\phi_p)}{p}\gamma_1\phi_p + \lambda(\phi_p)\gamma_0\right) l(T)$$

if $f^1_p \neq 0$, and it coincides with

$$\psi_E = (\nu(\phi_p, \delta_q)\delta_q + \frac{\lambda(\phi_p)}{p}\phi_p + \lambda(\phi_p)\gamma_0)l(T)$$

if $f^1_p = 0$. In the first case, by Proposition 8.29 we have that

$$\begin{array}{rcl} \delta_q\psi_E & = & (\nu(p\phi_p, \delta_q)\delta_q + \lambda(p\phi_p)p\phi_p^2 + \lambda(p\phi_p)\gamma_1\phi_p + \lambda(p\phi_p)\gamma_0)\delta_q l(T) \\ & = & (\nu(p\phi_p, \delta_q)\delta_q + \lambda(p\phi_p)p\phi_p^2 + \lambda(p\phi_p)\gamma_1\phi_p + \lambda(p\phi_p)\gamma_0)\psi^1_q \,. \end{array} \tag{124}$$

Similarly, if $f^1_p = 0$ we have

$$\begin{array}{rcl} \delta_q\psi_E & = & (\nu(p\phi_p, \delta_q)\delta_q + \lambda(p\phi_p)\phi_p + \lambda(p\phi_p)\gamma_0)\delta_q l(T) \\ & = & (\nu(p\phi_p, \delta_q)\delta_q + \lambda(p\phi_p)\phi_p + \lambda(p\phi_p)\gamma_0)\psi^1_q \,. \end{array} \tag{125}$$

Thus, if $\psi_E u = 0$, we have $\psi^1_q u$ a solution of

$$\nu(p\phi_p, \delta_q)\delta_q + \lambda(p\phi_p)p\phi_p^2 + \lambda(p\phi_p)\gamma_1\phi_p + \lambda(p\phi_p)\gamma_0$$

when $f^1_p \neq 0$, or a solution of

$$\nu(p\phi_p, \delta_q)\delta_q + \lambda(p\phi_p)\phi_p + \lambda(p\phi_p)\gamma_0$$

when $f^1_p = 0$.

Let $\alpha \in R$, and set

$$h := \int u_{a,\kappa,\alpha}\frac{dq}{q} \in qK[[q]]\,,$$

with $u_{a,\kappa,\alpha}$ as in (120) or (123), respectively. Note that

$$\delta_q h = u_{a,\kappa,\alpha}\,.$$

By Theorem 6.10, when $f^1_p \neq 0$ we get that

$$\begin{array}{rcl} \delta_q(\mu(0,\kappa)\alpha\kappa^{-1}q^\kappa) & = & \mu(0,\kappa)\alpha q^\kappa \\ & = & (\nu(p\phi_p, \delta_q)\delta_q + \lambda(p\phi_p)p\phi_p^2 + \lambda(p\phi_p)\gamma_1\phi_p + \lambda(p\phi_p)\gamma_0)u_{a,\kappa,\alpha} \\ & = & (\nu(p\phi_p, \delta_q)\delta_q + \lambda(p\phi_p)p\phi_p^2 + \lambda(p\phi_p)\gamma_1\phi_p + \lambda(p\phi_p)\gamma_0)\delta_q h \\ & = & \delta_q\left(\nu(\phi_p, \delta_q)\delta_q + \frac{\lambda(\phi_p)}{p}\phi_p^2 + \frac{\lambda(\phi_p)}{p}\gamma_1\phi_p + \lambda(\phi_p)\gamma_0\right) h \end{array} .$$

Hence

$$\begin{aligned}(126)\qquad &\left(\nu(\phi_p,\delta_q)\delta_q+\tfrac{\lambda(\phi_p)}{p}\phi_p^2+\tfrac{\lambda(\phi_p)}{p}\gamma_1\phi_p+\lambda(\phi_p)\gamma_0\right)h-\mu(0,\kappa)\alpha\kappa^{-1}q^{\kappa}\\ &\qquad\in R\cap q^{\pm 1}K[[q^{\pm 1}]]=0,\end{aligned}$$

and we have that

$$(127)\qquad \begin{array}{rcl}\lambda(\phi_p)\left(\frac{1}{p}\phi_p^2+\frac{1}{p}\gamma_1\phi_p+\gamma_0\right)h & = & -\nu(\phi_p,\delta_q)\delta_q h+\mu(0,\kappa)\alpha\kappa^{-1}q^{\kappa}\\ & = & -\nu(\phi_p,\delta_q)u_{a,\kappa,\alpha}+\mu(0,\kappa)\alpha\kappa^{-1}q^{\kappa}\\ & \in & q^{\pm 1}R[[q^{\pm 1}]]\,.\end{array}$$

By Lemma 2.6, we get

$$(128)\qquad \left(\frac{1}{p}\phi_p^2+\frac{1}{p}\gamma_1\phi_p+\gamma_0\right)h\in q^{\pm 1}R[[q^{\pm 1}]]\subset R[[q^{\pm 1}]]\,.$$

Similarly, when $f_p^1=0$, we get

$$(129)\qquad \left(\frac{1}{p}\phi_p+\gamma_0\right)h\in R[[q^{\pm 1}]]\,.$$

We claim that that if $f_p^1\neq 0$ we have

$$(130)\qquad \left(\frac{1}{p}\phi_p^2+\frac{1}{p}\gamma_1\phi_p+\gamma_0\right)l_E(q)\in R[[q^{\pm 1}]]\,,$$

where $l_E(q)\in K[[q^{\pm 1}]]$ is obtained from $l(T)=l_E(T)\in K[[T]]$ by substitution of $q^{\pm 1}$ for T, and $\phi_p: R[[q^{\pm 1}]]\to R[[q^{\pm 1}]]$ is defined by $\phi_p(q^{\pm 1})=q^{\pm p}$. In order to check that this holds, it is sufficient to check that

$$(131)\qquad (\phi_p^2+\gamma_1\phi_p+p\gamma_0)l_E(q^{\pm 1})\in pR[[q^{\pm 1}]]\,.$$

Now recall that

$$(132)\qquad (\phi_p^2+\gamma_1\phi_p+p\gamma_0)l_E(T)\in pR[[T]][\delta_pT,\delta_p^2T]\hat{}\,,$$

where, as usual, the mappings $R[[T]]\stackrel{\phi_p}{\to}R[[T]][\delta_pT]\hat{}\stackrel{\phi_p}{\to}R[[T]][\delta_pT,\delta_p^2T]\hat{}$ are defined by $\phi_p(T)=T^p+p\delta_pT$, $\phi_p(\delta_pT)=(\delta_pT)^p+p\delta_p^2T$. Taking the image (132) under the unique δ_p-ring homomorphism $R[[T]][\delta_pT,\delta_p^2T]\hat{}\to R[[q^{\pm 1}]]$ that sends T into $q^{\pm 1}$, we get equality (131), completing the verification that (130) holds.

Similarly, when $f_p^1=0$ we have that

$$(133)\qquad \left(\frac{1}{p}\phi_p+\gamma_0\right)l_E(q)\in R[[q^{\pm 1}]]\,.$$

If $f_p^1\neq 0$, then by (128), (130), and Hazewinkel's Functional Equation Lemma 2.5, it follows that $e(h)\in R[[q^{\pm 1}]]$. Since $e(h)\in q^{\pm 1}K[[q^{\pm 1}]]$, we get

$$u_{E,\kappa,\alpha}=e_E(h)\in q^{\pm 1}R[[q^{\pm 1}]]\,.$$

Similarly, if $f_p^1=0$, by (129) and (133), we get that

$$u_{E,\kappa,\alpha}\in q^{\pm 1}R[[q^{\pm 1}]]\,.$$

As in the proof of Lemma 7.6, we see that

$$\begin{array}{rcl}R & \to & R[[q^{\pm 1}]]\\ \alpha & \mapsto & u_{E,\kappa,\alpha}\end{array}$$

is pseudo δ_p-polynomial mapping. $\square$

We have the following diagonalization result.

Lemma 8.37. *Let $\kappa \in \mathbb{Z}\backslash p\mathbb{Z}$, and $\alpha \in R$. Then*

$$\begin{array}{rcl} \psi^1_q u_{E,\kappa,\alpha} & = & u_{a,\kappa,\alpha} \\ \psi_E u_{E,\kappa,\alpha} & = & \mu(0,\kappa)\alpha\kappa^{-1}q^{\kappa}. \end{array}$$

Proof. The first equality is clear. Now, by (126), if $f^1_p \neq 0$ we have that

$$\begin{array}{rcl} \psi_E u_{E,\kappa,\alpha} & = & \left(\nu(\phi_p,\delta_q)\delta_q + \frac{\lambda(\phi_p)}{p}\phi_p^2 + \frac{\lambda(\phi_p)}{p}\gamma_1\phi_p + \lambda(\phi_p)\gamma_0\right) l_E(e_E(h)) \\ & = & \mu(0,\kappa)\alpha\kappa^{-1}q^{\kappa}\,. \end{array}$$

A similar argument can be given in the case $f^1_p = 0$. □

Remark 8.38.

1) For $\kappa \in \mathbb{Z}\backslash p\mathbb{Z}$ we have

(134) $$u_{E,\kappa,\zeta^{\kappa}\alpha}(q) = u_{E,\kappa,\alpha}(\zeta q)$$

for all $\zeta \in \mu(R)$. So if $\alpha = \sum_{i=0}^{\infty} m_i\zeta_i^{\kappa}$, $\zeta_i \in \mu(R)$, $m_i \in \mathbb{Z}$, $v_p(m_i) \to \infty$, then

$$u_{E,\kappa,\alpha}(q) = \left[\sum_{i=0}^{\infty}\right][m_i](u_{E,\kappa,1}(\zeta_i q))\,.$$

If $f \in \mathbb{Z}\mu(R)\widehat{\ }$ is such that $(f^{[\kappa]})^{\sharp} = \alpha \in R$, then

$$u_{E,\kappa,\alpha} = f \star u_{E,\kappa,1}\,.$$

Note that $\{u_{E,\kappa,\alpha} \mid \alpha \in R\}$ is a $\mathbb{Z}\mu(R)\widehat{\ }$-module (under convolution). This module structure comes from an R-module structure (still denoted by $\star$) by base change via the morphism

$$\mathbb{Z}\mu(R)\widehat{\ } \stackrel{[\kappa]}{\to} \mathbb{Z}\mu(R)\widehat{\ } \stackrel{\sharp}{\to} R$$

(cf. (34)), and the R-module $\{u_{E,\kappa,\alpha} \mid \alpha \in R\}$ is free, with basis $u_{E,\kappa,1}$. Thus, for $g \in \mathbb{Z}\mu(R)\widehat{\ }$, $\beta = (g^{[\kappa]})^{\sharp}$, we have that

$$\beta \star u_{E,\kappa,\alpha} = g \star u_{E,\kappa,\alpha}\,.$$

In particular

$$u_{E,\kappa,\alpha} = \alpha \star u_{E,\kappa,1}\,.$$

2) We recall (see (31)) the natural group homomorphisms attached to ψ^1_q,

$$\begin{array}{rcl} B^0_{\kappa} : E(R[[q^{\pm 1}]]) & \to & R \\ B^0_{\kappa} u & = & \Gamma_{\kappa}\psi^1_q u \end{array}.$$

For $\kappa_1, \kappa_2 \in \mathbb{Z}\backslash p\mathbb{Z}$, we obtain that

(135) $$B^0_{\kappa_1} u_{E,\kappa_2,\alpha} = \Gamma_{\kappa_1} u^{\mu}_{a,\kappa_2,\alpha} = \alpha \cdot \delta_{\kappa_1\kappa_2}\,.$$

3) Assume that E is defined over $\mathbb{Z}_p$ and $f^1_p \neq 0$. Then $f^1_p \in \mathbb{Z}_p$, so $\gamma_0 = 1$. Also, by the Introduction of [6], $\gamma_1 \in \mathbb{Z}$. Thus, if in addition to $\alpha, \lambda \in \mathbb{Z}_{(p)}$ we have that the cubic defining E has coefficients in $\mathbb{Z}_{(p)}$, then $u_{E,\kappa,\alpha} \in \mathbb{Z}_{(p)}[[q^{\pm 1}]]$ for $\kappa \in \mathbb{Z}\backslash p\mathbb{Z}$.

4) Assume E is defined over $\mathbb{Z}_p$ and that $f^1_p = 0$. Then the Introduction in [6], γ_0 is an integer in a quadratic extension F of $\mathbb{Q}$. We view F as embedded into $\mathbb{Q}_p$, and set $\mathcal{O}_{(p)} := F \cap \mathbb{Z}_p$. Then, if in addition to $\alpha, \lambda \in \mathbb{Z}_{(p)}$ we have that the cubic defining E has coefficients in $\mathbb{Z}_{(p)}$, we obtain that $u_{E,\kappa,\alpha} \in \mathcal{O}_{(p)}[[q^{\pm 1}]]$ for $\kappa \in \mathbb{Z}\backslash p\mathbb{Z}$.

Theorem 8.39. *Let E be an elliptic curve over R, ψ_E be a non-degenerate $\{\delta_p,\delta_q\}$-character of E, and $\mathcal{U}_*$ be the corresponding groups of solutions. If $\mathcal{K}$ is the set of characteristic integers of ψ_E, and $u_{E,\kappa,\alpha}$ be the basic series, then we have that*

$$\mathcal{U}_{\pm 1} = \oplus_{\kappa\in\mathcal{K}_\pm}\{u_{E,\kappa,\alpha} \mid \alpha\in R\}\,,$$

where $\oplus$ stands for the internal direct sum. In particular, $\mathcal{U}_{\pm 1}$ are free R-modules under convolution, with bases $\{u_{E,\kappa,1} \mid \kappa\in\mathcal{K}_\pm\}$, respectively.

Proof. By Lemma 8.37, $u_{E,\kappa,\alpha}\in\mathcal{U}_1$. Conversely, if $u\in\mathcal{U}_1$, that is to say, if $u\in E(qR[[q]])$ and $\psi_E u=0$, we have that $\psi_q^1 u\in qR[[q]]$ is a solution of

$$\nu(p\phi_p,\delta_q)\delta_q+\lambda(p\phi_p)p\phi_p^2+\lambda(p\phi_p)\gamma_1\phi_p+\lambda(p\phi_p)\gamma_0\,,$$

or a solution of

$$\nu(p\phi_p,\delta_q)\delta_q+\lambda(p\phi_p)\phi_p+\lambda(p\phi_p)\gamma_0\,,$$

if $f_p^1\neq 0$ or $f_p^1=0$, respectively. Cf. (124) and (125), respectively. By Theorem 6.10, we have

$$\psi_q^1 u=\sum_{i=1}^s u^{\mu}_{a,\kappa_i,\alpha_i}$$

for some $\alpha_i\in R$, where $\mathcal{K}_+=\{\kappa_1,\dots,\kappa_s\}$. Therefore,

$$u=e_E\left(\int\sum_{i=1}^s u^{\mu}_{a,\kappa_i,\alpha_i}\frac{dq}{q}\right)=\sum_{i=1}^s u_{E,\kappa_i,\alpha_i}\,.$$

This representation is unique because of (135).

A similar argument works for $u\in\mathcal{U}_{-1}$. $\square$

Corollary 8.40. *Under the hypotheses of Theorem 8.39, let $u\in\mathcal{U}_\pm$. Then the following hold:*

1. *The series $\overline{\psi_q^1 u}\in k[[q^{\pm 1}]]$ is integral over $k[q^{\pm 1}]$, and the field extension $k(q)\subset k(q,\overline{\psi_q^1 u})$ is Abelian with Galois group killed by p.*
2. *If the characteristic polynomial of ψ_E is unmixed and $\mathcal{K}_\pm$ is short then u is transcendental over $K(q)$.*

Proof. Assertion 1 follows immediately by Theorem 8.39 and Lemmas 6.6 and 8.37, respectively. In order to check assertion 2, note that by the above results, $\psi_q^1 u$ is transcendental over $K(q)$. If u were algebraic over $K(q)$, the point

$$(T(q),W(q))=(u,u^3+a_4u^7+\cdots)$$

would have algebraic coordinates over $K(q)$. Hence, the same would be true about the point

$$(x(q),y(q))=\left(\frac{T(q)}{W(q)},-\frac{1}{W(q)}\right)\,,$$

and therefore, $\psi_q^1 u=\delta_q x(q)/y(q)$ would be algebraic over $K(q)$, a contradiction. Thus, u is transcendental over $K(q)$, and assertion 2 is proved. $\square$

Corollary 8.41. *Under the hypotheses of Theorem 8.39, the maps $B^0_\pm:\mathcal{U}_{\pm 1}\to R^{\rho_\pm}$ are R-module isomorphisms. Furthermore, for any $u\in\mathcal{U}_{\pm 1}$, we have*

$$u=\sum_{\kappa\in\mathcal{K}_\pm}(B^0_\kappa u)\star u_{E,\kappa,1}\,.$$

In particular the "boundary value problem at $q^{\pm 1}=0$" is well posed.

Next we address the "boundary value problem at $q\neq 0$" for ψ_E.

If E is an elliptic curve over R, we denote by $E(pR)$ the kernel of the reduction modulo p map $E(R)\to E(k)$. As usual, the group $E(pR)$ will be identified with pR via the bijection

$$\begin{array}{rcl} pR & \to & E(pR) \\ pa & \mapsto & (pa,(pa)^3+a_4(pa)^7+\cdots) \end{array} .$$

We denote by $E'(k)$ the group of all points in $E(k)$ of order prime to p. And we denote by $E'(R)$ the subgroup of all points in $E(R)$ whose image in $E(k)$ lie in $E'(k)$. There is a split exact sequence $0\to E(pR)\to E'(R)\to E'(k)\to 0$, hence $E'(R)=E(pR)\oplus E'(R)_{tors}$.

Corollary 8.42. *Under the hypotheses of Theorem* 8.39*, and letting*

$$\mathcal{U}'_+ := E'(R)_{tors}\cdot \mathcal{U}_1 \subset \mathcal{U}_+\, ,$$

the following hold:

1) *Assume $\mathcal{K}_+=\{\kappa\}$. For any $q_0\in p^\nu R^\times$ with $\nu\geq 1$, and any $g\in p^{\nu\kappa}R\subset pR=E(pR)$ there exists a unique $u\in\mathcal{U}_1$ such that $u(q_0)=g$.*
2) *Assume $\mathcal{K}_+=\{1\}$. Then for any $q_0\in pR^\times$ and any $g\in E'(R)$ there exists a unique $u\in\mathcal{U}'_+$ such that $u(q_0)=g$.*

Proof. The first assertion follows exactly as in the case of $\mathbb{G}_m$; cf. Corollary 7.13.

In order to check the second assertion, note that we can write g uniquely as $g=g_1+g_2$, where $g_1\in E'(R)_{tors}$ and $g_2\in E(pR)$. By the first part, there exists $u_2\in\mathcal{U}_1$ such that $u_2(q_0)=g_2$. We set $u=g_1+u_2$. Then, $u(q_0)=g$.

The uniqueness of u is also checked easily. □

The following Corollary is concerned with the inhomogeneous equation $\psi_E u=\varphi$, and it is an immediate consequence of Lemmas 8.37, 8.41, and 6.6.

Corollary 8.43. *Let ψ_E be a non-degenerate $\{\delta_p,\delta_q\}$-character of E, and let $\varphi\in q^{\pm1}R[[q^{\pm1}]]$ be a series whose support is contained in the set $\mathcal{K}'$ of totally non-characteristic integers of ψ_E. Then the following hold:*

(1) *The equation $\psi_E u=\varphi$ has a unique solution $u\in E(q^{\pm1}R[[q^{\pm1}]])$ such that $\psi_q^1 u$ has support disjoint from the set $\mathcal{K}$ of characteristic integers.*
(2) *If $\bar{\varphi}\in k[q^{\pm1}]$ the series $\overline{\psi_q^1 u}\in k[[q^{\pm1}]]$ is integral over $k[q^{\pm1}]$ and the field extension $k(q)\subset k(q,\overline{\psi_q^1 u})$ is Abelian with Galois group killed by p.*
(3) *If the characteristic polynomial of ψ_E is unmixed and the support of φ is short then u is transcendental over $K(q)$.*

Remark 8.44. Corollary 8.42 implies that if $\mathcal{K}_+=\{1\}$ and $q_0\in pR^\times$, the group homomorphism

$$\begin{array}{rcl} S_{q_0}: E'(R)_{tors}\times R & \to & E'(R) \\ (P,\alpha) & \mapsto & P+u_{E,1,\alpha}(q_0) \end{array}$$

is an isomorphism. So for any $q_1,q_2\in pR^\times$, we have an isomorphism

$$S_{q_1,q_2} := S_{q_2}\circ S_{q_1}^{-1} : E'(R)\to E'(R)\, .$$

The latter map should be viewed as the "propagator" attached to ψ_E.

As in the case of $\mathbb{G}_a$ and $\mathbb{G}_m$, respectively, if $\zeta \in \mu(R)$ and $q_0 \in pR^\times$, then, by (134),

$$S_{\zeta q_0} = S_{q_0} \circ M_\zeta$$

where

$$\begin{array}{rcl} M_\zeta : E'(R)_{tors} \times R & \rightarrow & E'(R)_{tors} \times R \\ M_\zeta(P, \alpha) & := & (P, \zeta\alpha) \end{array}$$

Hence, for $\zeta_1, \zeta_2 \in \mu(R)$, we obtain that

$$S_{\zeta_1 q_0, \zeta_2 q_0} = S_{q_0} \circ M_{\zeta_2/\zeta_1} \circ S_{q_0}^{-1}.$$

In particular,

$$S_{q_0, \zeta_1\zeta_2 q_0} = S_{q_0, \zeta_2 q_0} \circ S_{q_0, \zeta_1 q_0},$$

which can be interpreted as a (weak) “Huygens principle.”

References

[1] M. Barcau, *Isogeny covariant differential modular forms and the space of elliptic curves up to isogeny*, Compositio Math., 137 (2003), pp. 237-273.

[2] A. Buium, *Intersections in jet spaces and a conjecture of S.Lang*, Ann. of Math., 136 (1992), pp. 557-567.

[3] A. Buium, *Differential Algebra and Diophantine Geometry*, Hermann, Paris, 1994.

[4] A. Buium, *Geometry of differential polynomial functions I: algebraic groups*, Amer. J. Math., 115 (1993), pp. 1385-1444.

[5] A. Buium, *Differential characters of Abelian varieties over p-adic fields*, Invent. Math., 122 (1995), pp. 309-340.

[6] A. Buium, *Differential characters and characteristic polynomial of Frobenius*, J. reine angew. Math., 485 (1997), pp. 209-219.

[7] A. Buium, *Differential modular forms*, J. reine angew. Math., 520 (2000), pp. 95-167.

[8] A. Buium, *Geometry of Fermat adeles*, Trans. Amer. Math. Soc., 357 (2004), pp. 901-964.

[9] A. Buium, *Arithmetic Differential Equations*, Math. Surveys and Monographs 118, AMS, 2005.

[10] B. Dwork, *Lectures on p-adic Differential Equations*, Grundlehren der math. Wiss. 253, Springer, 1982.

[11] P. Cassidy, *Differential algebraic groups*, Amer.J.Math. 94 (1972), 891-954.

[12] Yu.V. Egorov, A.I. Komech & M.A. Shubin, *Elements of the Modern Theory of Partial Differential Equations*, Springer, 1999.

[13] G.B. Folland, *Introduction to Partial Differential Equations*, Princeton University Press, 1995.

[14] E. Frenkel & D. Ben-Zvi, *Vertex Algebras and Algebraic Curves*, Math. Surveys and Monographs 88, AMS, 2004.

[15] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Math., Springer Verlag, 1977.

[16] M. Hazewinkel, *On formal groups. The functional equation lemma and some of its applications*, Astérisque, 63 (1979), pp. 73-82.

[17] L. Hörmander, *The Analysis of Linear Partial Differential Operators, Volumes I and II*, Grundlehren der math. Wiss., Springer Verlag, 1983.

[18] C. Hurlburt, *Isogeny covariant differential modular forms modulo p*, Compositio Math., 128 (2001), pp. 17-34.

[19] N. Koblitz, *p-adic numbers, p-adic analysis, and Zeta Functions*, GTM 58, Springer Verlag, 1984.

[20] E.R. Kolchin, *Differential Algebra and Algebraic Groups*, Academic Press , New York, 1973.

[21] N. Kurokawa, H. Ochiai & M. Wakayama, *Absolute derivations and zeta functions*, Documenta Math., Extra Volume Kato (2003), pp. 565-584.

[22] Yu.I. Manin, *Algebraic curves over fields with differentiation*, Izv. Akad. Nauk SSSR, Ser. Mat. 22 (1958), pp. 737-756 = AMS translations Series 2, 37 (1964), pp. 59-78.

[23] Yu.I. Manin, *Lectures on zeta functions and motives (according to Deninger and Kurokawa)*, Astérisque 228 (1995), pp. 121-163.

[24] P.J. Olver, *Applications of Lie Groups to Differential Equations*, GTM 107, Springer Verlag, 2000.
[25] J. Rauch, *Partial differential equations*, GTM 128, Springer, 1991.
[26] J.F. Ritt, *Differential Algebra*, AMS Coll. Pub 33, AMS, 1950.
[27] J-P. Serre, *Algebraic groups and class fields*, GTM 117, Springer, 1988.
[28] J.H. Silverman, *The Arithmetic of Elliptic Curves*, Springer Verlag, Berlin, New York, 1986.
[29] J.H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, GTM 151, Springer Verlag, 1994.
[30] C. Soulé, *Les variétés sur le corps à un élément*, Moscow Math. J., 4 (2004), pp. 217-244.
[31] S. Uchiyama, *On the Thue-Siegel-Roth theorem, III*, Proc. Japan Acad. 26 (1960), 1-2.
[32] V.S. Vladimirov, I.V. Volovich & E. I. Zelenov, *p-adic Analysis and Mathematical Physics*, World Scientific, 1994.

University of New Mexico, Albuquerque, NM 87131
E-mail address: buium@math.unm.edu, santiago@math.unm.edu